SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 4

Fall 2003

IV. Smooth mappings

IV.1 : Linear approximations

Problems from Edwards, §§ II.1, *p.* 75

2.6 Use the derivative approximation to evaluate the following:

(a) $[(3.02)^2 + (1.97)^2 + (5.98)^2]$

SOLUTION.

Take $f(x, y, z) = x^2 + y^2 + z^2$, so that $\nabla f(v) = 2v$. The general approximation rule is

$$h(v + \Delta v) \approx h(v) + \langle \nabla h(v), \Delta v \rangle$$

and in this special case $h = f$, $v = (3, 2, 6)$ and $\Delta v = (0, 02, -0.03, -0.02)$. Thus the formula specializes to

$$[(3.02)^2 + (1.97)^2 + (5.98)^2] \approx [3^2 + 2^2 + 6^2] + \langle (6, 4, 12)\,,\ (0, 02, -0.03, -0.02) \rangle$$

which simplifies to 48.76; for the sake of comparison, we note that the actual value is 48.7617 .■

(b) $(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2 \right)$

SOLUTION.

Take $f(x, y) = x^2 - y^2$, so that $\nabla f(x, y) = (2xe^{x^2 - y^2}, -2ye^{x^2 - y^2})$. This provides some of the substitutions needed in the general approximation rule described above. The remaining pieces are $v = (1, 1)$, $f!, 1) = 1$ and $\Delta v = (0.1, -0.1)$. Thus the formula specializes to

$$(e^4)^{1/10} = \exp\left((1.1)^2 - (0.9)^2 \right) \approx e^{1^2 - 1^2} + 4 \cdot (0.1) \cdot e^{1^2 - 1^2} = 1.4$$

For the sake of comparison, we note that the actual value is 1.4918247 to seven decimal places.■

2.8 Let $f : \mathbf{R}^n \to \mathbf{R}$ be differentiable. If $f(0) = 0$ and $f(tx) = t f(x)$ for all t and x prove that $f(x) = \langle \nabla f(0), x \rangle$ for all x; *i.e.*, f is linear. Consequently, any nonlinear function g satisfying the conditions $g(0) = 0$ and $g(tx) = t\, g(x)$ for all t and x is not differentiable although it has directional derivatives in all directions at the origin (why?).

SOLUTION.

Let $x \in \mathbf{R}^n$ be an arbitrary vector; then

$$[Df(0)](x) \quad = \quad \lim_{t \to 0} \frac{1}{t} \cdot (\, f(tx) - f(0)\,) \quad = \quad \lim_{t \to 0} f(x)$$

and the latter is just $f(x)$. Since the left hand side is a linear function of u the same is true of the right hand side. But this means that $f(x) = \langle \nabla f(0), x \rangle$ for all x.

If g is a nonlinear function satisfying the conditions $g(0) = 0$ and $g(tx) = tg(x)$ for all t and x, the right hand side of the displayed formula shows that g has directional derivatives in all directions through the origin, but if it were differentiable it would have to be linear. Therefore g cannot be linear.■

Problems from Edwards, §§ II.2, pp. 88 – 90

3.3 let F, $G : \mathbf{R}^n \to \mathbf{R}^n$ be differentiable mappings and let $h : \mathbf{R}^n \to \mathbf{R}$ be a differentiable function. Given u, $v \in \mathbf{R}^n$ show the following:

(a) $D_u(F+G) = D_u(F) + D_u(G)$

(b) $D_{u+v}(F) = D_u(F) + D_v(F)$

(c) $D_u(hF) = D_u(h) \cdot F + h \cdot D_u(F)$

SOLUTION.

On page 65 of Edwards, $D_u F(a)$ is defined to be the ordinary derivative of $g(t) = F(a+tu)$ at $t = 0$. With this definition the verifications of **(a)** and **(c)** become exercises in the differentiation of vector valued functions of a single real variable; we shall not give the details. Part **(b)** requires Theorem 2.1 on pages 68–69 of Edwards. The latter proves that $D_w f(x) = [Df(x)](w)$. The sum formula is then an immediate consequence of the linearity of $Df(x)$.■

3.16 Define $f : \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) \quad = \quad \frac{xy(x^2-y^2)}{x^2+y^2}$$

for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

(a) Show that $D_1 f(0,y) = -y$ and $D_2 f(x,0) = x$ for all x and y.

SOLUTION.

We have

$$D_1 f(0,y) \quad = \quad \lim_{t\to 0} \frac{1}{t}\,\frac{ty(t^2-y^2)}{t^2+y^2} \quad = \quad \lim_{t\to 0} \frac{yt^2-y^3}{t^2+y^2}$$

for all y. If $y \neq 0$ then this limit can easily be evaluated as $-y$. On the other hand if $y = 0$ then we have $f(x,0) = 0$ so that the first partial is $0 = -y$.

Similarly,

$$D_2 f(x,0) \quad = \quad \lim_{t\to 0} \frac{1}{t}\,\frac{xt(x^2-t^2)}{x^2+t^2} \quad = \quad \lim_{t\to 0} \frac{x(x^2-t^2)}{x^2+t^2}$$

for all x. If $x \neq 0$ then this limit can easily be evaluated as x. On the other hand if $x = 0$ then we have $f(0,y) = 0$ so that the first partial is $0 = x$.■

(b) Conclude that $D_1 D_2 f(0,0)$ and $D_2 D_1 f(0,0)$ exist but are not equal.

SOLUTION.

Since $D_2 f(x,0) = x$ it follows that $D_1 D_2 f(0,0) = 1$, and since $D_1 f(0,y) = -y$ it follows that $D_2 D_1 f(0,0) = -1$.■

FOOTNOTE.

One easy way to see the continuity of f at the origin is to write it in terms of polar coordinate; in these terms the value of the function is

$$\frac{r^2 \sin 4\theta}{4}$$

(verify this), and continuity at the origin is clear from this reformulation.■

IV.2 : Properties of smooth functions

Problems from Edwards, § II.3, *pp.* 88 – 90

3.4 Show that each of the following is a solution of the *heat equation*

$$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2}$$

(where k is a constant):

(a) $\exp(-k^2 a^2 t) \sin ax$

SOLUTION.

Direct computations show that both sides of the partial differential equation are equal to $-k^2 a^2 \exp(-k^2 a^2 t) \sin ax$.■

(b) $\exp(-x^2/4k^2 t)/\sqrt{t}$

SOLUTION.

Direct but significantly more tedious computations show that both sides of the partial differential equation are equal to

$$\frac{x^2 - 2k^2}{4k^2 t^{3/2}} \exp(-x^2/4k^2 t) \ .$$

3.11 **(a)** If $f(x) = g(\rho)$ where $\rho = |x|$ and the number n of variables is at least 3, show that

$$\nabla^2 f = \frac{n-1}{\rho} g'(\rho) + g''(\rho)$$

for $x \neq 0$.

SOLUTION.

We need to compute the second partials with respect to each variable x_i and add them up. By the Chain Rule

$$\frac{\partial f}{\partial x_i} = g'(\rho) \frac{\partial \rho}{\partial x_i}$$

and the second factor of the right hand side is equal to

$$\frac{x_i}{\rho}$$

because $\rho = \left(\sum_j x_j^2\right)^{1/2}$. We must next differentiate this function

$$\frac{x_i\, g'(\rho)}{\rho}$$

once again with respect to x_i and see what happens.

$$\frac{\partial^2 f}{\partial x_i^2} \;=\; \frac{\partial}{\partial x_i}\left(\frac{x_i\, g'(\rho)}{\rho}\right) \;=\; x_i\,\frac{\partial}{\partial x_i}\left(\frac{g'(\rho)}{\rho}\right) \;+\; \frac{g'(\rho)}{\rho} \;=$$

$$\frac{d}{d\rho}\left(\frac{g'(\rho)}{\rho}\right)\frac{x_i^2}{\rho} \;+\; \frac{g'(\rho)}{\rho} \;=\; \left(\frac{g''(\rho)}{\rho} - \frac{g'(\rho)}{\rho^2}\right)\frac{x_i^2}{\rho} \;+\; \frac{g'(\rho)}{\rho}$$

If we add these expressions over all i such that $1 \leq i \leq n$, we obtain the Laplacian of f. Since

$$\sum_{i-1}^{n} \frac{x_i^2}{\rho} = \rho$$

(recall that $\sum_i x_i^2 = \rho^2$) the sum of all the second partial derivatives that gives the Laplacian is equal to

$$\sum_{i-1}^{n}\left(\frac{g''(\rho)}{\rho} - \frac{g'(\rho)}{\rho^2}\right)\frac{x_i^2}{\rho} \;+\; \frac{g'(\rho)}{\rho}$$

which simplifies to

$$g''(\rho) - \frac{g'(\rho)}{\rho} + \frac{n\, g'(\rho)}{\rho}$$

and the latter simplifies further to the expression at the end of the exercise.■

(b) Using the formula displayed above, prove that if $\nabla^2 f = 0$ then

$$f(x) \;=\; \frac{a}{|x|^{n-2}} + b$$

where $x \neq 0$ and a and b are constants.

SOLUTION.

By the formula from the preceding exercise we have

$$\frac{n-1}{\rho}\, g'(\rho) \;+\; g''(\rho) \;=\; 0$$

which reduces to a separable first order differential equation in g' whose associated general solution for $g(\rho) = f(x)$ has the indicated form.

3.12 Verify that the functions $r^n \cos^n \theta$ and $r^n \sin^n \theta$ satisfy the 2-dimensional Laplace equation in polar coordinates. [Exercise 3.9 on the same page gives the formula for the Laplacian in polar coordinates.]

SOLUTION.

The polar form of the Laplacian is

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 g}{\partial \theta^2} + \frac{1}{r}\frac{\partial g}{\partial r} .$$

If we substitute $g(r,\theta) = r^n \cos^n \theta$ into this expression we obtain

$$n(n-1)\, r^{n-2} \cos n\theta \;-\; n^2 r^{n-2} \cos n\theta \;+\; n r^{n-2} \cos n\theta$$

which is equal to 0.■

3.13 If $f(x,y,z) = g(t-\rho/c)\,/\,\rho$ where $\rho = (x^2+y^2+x^2)^{1/2}$ and c is a constant, show that f satisfies the 3-dimensional *wave equation*

$$\nabla^2 f \;=\; \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} .$$

SOLUTION.

The idea again is to compute both sides explicitly and note that they are equal. For this purpose it will be helpful to use the formula for the Laplacian in spherical coordinates that is given in Exercise 3.10(b) on the same page as the problem we are working. If we write $f(x,y,z)$ as $F(\rho,\theta,\phi)$ and F does not depend upon θ or ϕ (as in our case), the Laplacian formula in spherical coordinates reduces to

$$\nabla^2 f \;=\; \frac{\partial^2 F}{\partial \rho^2} \;+\; \frac{2}{\rho}\frac{\partial F}{\partial \rho} .$$

Computing the right hand side of the wave equation for our choice of f is trivial:

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} \;=\; \frac{1}{c^2}\frac{g''(t-\rho/c)}{\rho}$$

What about the left hand side? Of course the first step is to compute the partial derivative with respect to ρ:

$$\frac{\partial F}{\partial \rho} \;=\; \frac{1}{\rho}\frac{(-1)}{c}\, g'(t-\rho/c) \;-\; \frac{1}{\rho^2}\, g(t-\rho/c)$$

If we take partial derivatives with respect to ρ again we obtain the following:

$$\frac{\partial^2 F}{\partial \rho^2} \;=\; \frac{1}{\rho}\frac{1}{c^2}\, g''(t-\rho/c) + \frac{1}{\rho^2}\frac{1}{c}\, g'(t-\rho/c) - \frac{1}{\rho}\frac{(-1)}{c}\, g'(t-\rho/c) + \frac{2}{\rho^3}\, g(t-\rho/c)$$

If we substitute these into the expression for the Laplacian in spherical coordinates we obtain the same function that we obtained for the right hand side of the wave equation.■

3.14 The following shows the hazards of denoting functions by real variables. Let $w = f(x,y,z)$ and $z = g(x,y)$. Then

$$\frac{\partial w}{\partial x} \;=\; \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x} \;=\; \frac{\partial w}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}$$

because the partials of x and y with respect to x are 1 and 0 respectively. Therefore

$$\frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0$$

But if $w = x+y+z$ and $z = x+y$ then the expression on the left hand side is $1 \cdot 1 = 1$, so that $0 = 1$. Where is the mistake?

SOLUTION.

Since the point of this fallacy is to show the need to write things down less casually, we should begin by doing so. The symbol w is actually being used for two separate functions; namely, $f(x,y,z)$ and $B(x,y) = f(x,y,\,g(x,y)\,)$. The application of the Chain Rule in the first line then becomes

$$\frac{\partial B}{\partial x} \quad = \quad \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}$$

which is consistent with the previous displayed line. The mistake leading to the fallacy is that one cannot assume that the partial derivatives of B and f with respect to x are equal; these are two separate functions, and in fact the given example illustrates this fact very clearly.∎

Problems from Edwards, §§ III.2, p. 180

2.1 Let α and β be norms on $\mathbf{R}^m$ and $\mathbf{R}^n$ respectively. Prove that $\gamma_0(x,y) = \alpha(x) + \beta(y)$ and $\gamma_1(x,y) = \max\,(\alpha(x),\,\beta(y)\,)$ define norms on $\mathbf{R}^{m+n} \cong \mathbf{R}^m \times \mathbf{R}^n$.

SOLUTION.

In the section on Cartesian products in the course notes we showed that functions of this sort defined metrics on a product. The proof that the functions described here are norms is similar.∎

2.8 Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a $1-1$ linear mapping. Prove that there is an $\varepsilon > 0$ such that if $S : \mathbf{R}^n \to \mathbf{R}^m$ is linear and satisfies $\|S - T\| < \varepsilon$, then S is also $1-1$. [The wording has been changed significantly to allow more flexibility in proving the main statement of the exercise.]

SOLUTION.

By the results on equivalences of norms, it suffices to prove the result for any norm on the space of $m \times n$ matrices. For the sake of definiteness we shall use the norm associated to the standard inner product on the space of all $m \times n$ matrices viewed as $\mathbf{R}^{mn}$. Denote this norm by $|...|_2$.

Suppose that A is an $m \times n$ matrix representing T; it is enough to show that all matrices sufficiently close to A have rank n if (as is the case here) A has rank n. Since the rank of A is n, one can choose n rows from A to obtain an $n \times n$ matrix $\varphi(A)$ that is invertible. We know that the set of invertible square matrices is open, so there is a $\delta > 0$ such that $|C - \varphi(A)|_2 < \delta$ implies that C is invertible. Since $|\varphi(B) - \varphi(A)|_2 \leq |B - A|_2$ this implies that the rank of B is at least n if $|B - A|_2 < \delta$. Since the rank of B is at most n, it follows that the rank is exactly n under the given condition and hence that the linear transformation associated to B is 1–1.∎

Additional exercises

1. **(a)** Suppose that X and Y are subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively and that $f : X \to Y$ and $g : Y \to \mathbf{R}^p$ are maps that satisfy Lipschitz conditions. Prove that the composite $g \circ f$ also satisfies a Lipschitz condition.

SOLUTION.

`To be inserted.`

(b) Suppose that $X \subset \mathbf{R}^n$, and let $f,\, g : X \to \mathbf{R}^m$ and $h : X \to \mathbf{R}$ satisfy Lipschitz conditions. Prove that $f + g$ satisfies a Lipschitz condition and if X is compact then $h \cdot f$ also satisfies a Lipschitz condition. If $h > 0$ and X is compact, does $1/h$ satisfy a Lipschitz condition? Prove this or give a counterexample.

SOLUTION.

Let A and B be Lipschitz constants for f and g respectively. Furthermore, let C be a Lipschitz constant for h.

We verify first that $f + g$ satisfies a Lipschitz condition:

$$|[f+g](x) - [f+g](x')| = |f(x) + g(x) - f(x') - g(x')| \;\leq\; |f(x) - f(x')| + |g(x) - g(x')| \;\leq$$

$$A \cdot |x - x'| + B \cdot |x - x'| \;=\; (A + B)\, |x - x'|$$

In order to prove that $h \cdot f$ satisfies a Lipschitz condition we need to use the upper bounds for $|f|$ and $|h|$ which are guaranteed by compactness. Call these bounds P and Q respectively. We then have that

$$|h(x)f(x) - h(x')f(x')| \;=\; |h(x)f(x) - h(x)f(x') + h(x)f(x') - h(x')f(x')| \;\leq$$

$$|h(x)f(x) - h(x)f(x')| + |h(x)f(x') - h(x')f(x')| \;\leq\; Q \cdot |f(x) - f(x')| + P \cdot |h(x) - h(x')| \;\leq$$

$$QA\,|x - x'| + PC\,|x - x'| \;=\; (QA + PC)\,|x - x'|$$

and hence the product satisfies a Lipschitz condition.

Finally, $1/h$ does satisfy a Lipschitz condition, and here is the proof: In addition to the preceding let $\mathbf{m}$ be the minimum value of $|h|$ on X. Then we have

$$\left| \frac{1}{h(x)} \;-\; \frac{1}{h(x')} \right| =$$

$$\frac{|h(x') \;-\; h(x)|}{|h(x) \cdot h(x')|} \;.$$

The denominator is at least $\mathbf{m}^2$, and the numerator is at most $C \cdot |x - x'|$, and therefore $C\mathbf{m}^{-2}$ is a Lipschitz constant for $1/h$.∎

(c) Suppose that $X \subset \mathbf{R}^n$, and let $f : X \to \mathbf{R}^m$ be given. Prove that f satisfies a Lipschitz condition if and only if all of its coordinate functions do.

SOLUTION.

$(\Longrightarrow)$ If f satisfies a Lipschitz condition with Lipschitz constant A and f_i is the $i^{\rm th}$ coordinate function, then

$$|f_i(x) - f_i(y)| \;\leq\; |f(x) - f(y)| \;\leq\; A \cdot |x - y|$$

for all x and y.∎

$(\Longleftarrow)$ If each coordinate function f_i satisfies a Lipschitz condition, then one can write $f = \sum_i f_i\, \mathbf{e}_i$ where $\mathbf{e}_i$ is the standard $i^{\rm th}$ unit vector; therefore the conclusion follows from the first part of the exercise and finite induction.∎

2. In the notation of the preceding exercise, suppose that $X = A \cup B$ and that f is continuous and satisfies Lipschitz conditions on A and B as well as on an open neighborhood of $A \cap B$. Does f satisfy a Lipschitz condition on $A \cup B$? Prove this or give a counterexample. What happens if we assume A and B are compact? Justify your answer.

SOLUTION.

The answer to the first question is no. Consider the function on $\mathbf{R} - \{0\}$ that is 1 for positive numbers and -1 for negative numbers. This satisfies a Lipschitz condition on A and B as well as an open neighborhood of $A \cap B = \emptyset$. However, if we take x and x' to be $\pm 1/n$ then $|f(x) - f(x')| = 2$ while $|x - x'| = 2/n$, and hence any Lipschitz constant for f on $A \cup B$ would have to be at least n for every positive integer n. Therefore f does not satisfy a Lipschitz condition on $A \cup B$.

The answer to the second question is yes. Let U be an open neighborhood of $A \cap B$ on which f satisfies a Lipschitz condition, and let K_0 be the associated Lipschitz constant. Similarly, let K_1 and K_2 be Lipschitz constants for f on A and B respectively.

Formally, there are initially 64 cases to consider depending upon whether or not u lies in A, B or U (a total of 8 possibilities), and likewise for v. However, many of the formal possibilities are inconsistent with the conditions that u and v belong to $A \cup B$. In particular, we cannot have $u \notin A$ and $u \notin B$ and we cannot have $u \in A$, $u \in B$, but $u \notin U \supset A \cap B$. This brings us down to 5 possibilities each for u and v. Here is the list of possibilities for u:

[$u1$] $u \notin A$, $u \in B$ and $u \notin U$
[$u2$] $u \notin A$, $u \in B$ and $u \in U$
[$u3$] $u \in A$, $u \notin B$ and $u \notin U$
[$u4$] $u \in A$, $u \notin B$ and $u \in U$
[$u5$] $u \in A$, $u \in B$ and $u \in U$

Of course, there is a similar list for v, and the set of all 25 possibilities is given by taking one from each list.

Fortunately, separate arguments are not needed for each of the 25 cases. In particular, the four cases obtained by combining the first two possibilities for u and v involve situations where both points lie in B and therefore $|f(u) - f(v)| \leq K_2\,|u - v|$. Likewise, the nine cases obtained by combining the last three possibilities for u and v involve situations where both points lie in A and therefore $|f(u) - f(v)| \leq K_1\,|u - v|$.

This leaves us with twelve cases; six are given by taking one of the first two possibilities for u and one of the last three possibilities for v, and six are given by switching the roles of u and v. If we do the latter, we are down to six cases.

The cases $[u2] + [v2]$ and $[u2] + [v3]$ are situations where both point belong to U and therefore $|f(u) - f(v)| \leq K_0\,|u - v|$. In each of the remaining cases, at least one of u or v does not belong to U. Consider the continuous function on

$$\big((A \cup B) \times (A \cup B)\big) - U \times U$$

defined by the quotient

$$\frac{|f(u) - f(v)|}{|u - v|} \, .$$

Since the domain of this function is compact, it has a maximum value K_3, and thus we have $|f(u) - f(v)| \leq K_3\,|u - v|$ if (u, v) lies in the set described above.

If we take K to be the largest of the numbers K_i for $0 \leq i \leq 3$, then K will be a Lipschitz constant for f on $A \cup B$.■

IV.3 : Inverse Function Theorem

Problems from Edwards, §§ III.3, pp. 194 – 196

3.1. Show that

$$f(x,y) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

is locally invertible near every point except the origin. Compute the inverse explicitly.

SOLUTION.

Since the problem asks for an explicit computation of the inverse, one reasonable way to start is to see what happens if one tries to solve the system of equations given in vector form by

$$(u,v) \quad = \quad \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

(this is easier than writing out two separate equations for u and v in terms of x and y). Next, think about what this map does geometrically. Given a nonzero point in the plane, it sends this point to a positive multiple of itself; if as usual we write $r^2 = x^2 + y^2$, the exact multiple is $1/r^2$. Formally one can see this by verifying the identity

$$u^2 + v^2 \quad = \quad \frac{1}{x^2+y^2} \ .$$

Using this formula it follows immediately that f maps the nonzero points of the plane to themselves in a 1–1 onto fashion and is equal to its own inverse. Since f is a $\mathbf{C}^1$ function, it follows in particular that the function is *globally* invertible on the set of nonzero points of the plane.■

3.3. Consider the map : $\mathbf{R}^3 \to \mathbf{R}^3$ defined by $f(x,y,z) = (x, y^3, z^5)$. Note that f has a global inverse g despite the fact that $Df(0)$ is not invertible. What does this imply about the differentiability of g at 0?

SOLUTION.

The map cannot be differentiable at the origin, for if it were then it would be an inverse to $Df(0)$ and the latter does not have an inverse. Of course the global continuous inverse to f is the function $g(u,v,w) = (u, v^{1/3}, w^{1/5})$.■

3.4. Show that the mapping $(u,v,w) : \mathbf{R}^3 \to \mathbf{R}^3$ defined by $u = x + e^y$, $v = y + e^z$ and $w = z + e^x$ is everywhere locally invertible.

SOLUTION.

Compute the Jacobian:

$$\begin{vmatrix} 1 & e^y & 0 \\ 0 & 1 & e^z \\ e^x & 0 & 1 \end{vmatrix} \quad = \quad 1 + e^{x+y+z} \quad \neq \quad 0 \ .■$$

3.14. Let $f : \mathbf{R}^3_{\mathbf{x}} \to \mathbf{R}^3_{\mathbf{y}}$ and $g : \mathbf{R}^3_{\mathbf{y}} \to \mathbf{R}^3_{\mathbf{x}}$ be $\mathbf{C}^1$ inverse functions. Show that

$$\frac{\partial g_1}{\partial y_1} = \frac{1}{J}\,\frac{\partial(f_2,f_3)}{\partial(x_2,x_3)}, \qquad J = \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)}$$

and obtain similar formulas for the other derivatives of coordinate functions of g.

SOLUTION.

By construction Dg and Df are 3×3 matrices that are inverse to each other. By Cramer's Rule, if B and A are mutually inverse 3×3 matrices, then

$$b_{1,1} = \frac{1}{\det A} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix}$$

and the formula for the partial derivative as a quotient ot two Jacobians follows immediately from this. Similar considerations hold for all the partial derivatives of coordinate functions; since writing things out would make the solution about eight times longer and the details are mechanical, we shall not do so.■

Additional exercises

0. Prove that $F(x,y) = (e^x + y,\, x - y)$ defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathbf{C}^\infty$ inverse.

SOLUTION.

`To be inserted.`

1. Prove that $F(x,y) = (xe^y + y,\, xe^y - y)$ defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^2$ with a $\mathbf{C}^\infty$ inverse.

SOLUTION.

By the Inverse Function Theorem it suffices to show that F is 1–1 onto $\mathbf{C}^\infty$ and has a nonzero Jacobian everywhere. The $\mathbf{C}^\infty$ condition is immediate, and one can compute the Jacobian directly:

$$\begin{vmatrix} e^y & x\,e^y + 1 \\ e^y & x\,e^y - 1 \end{vmatrix} \quad = \quad -2e^y \quad \neq \quad 0$$

To show that f is 1–1 onto we need to show that for each choice of u and v there is a unique solution to the system of equations

$$u \;=\; xe^y + y, \qquad v \;=\; xe^y - y\ .$$

Here is an elementary way of doing so. Subtacting the second equation from the first shows that $y = (u - v)/2$, and adding the two equations together yields

$$2\,x\,e^y \;=\; u + v\ .$$

Since we can solve uniquely for y, this equation shows that we can also solve uniquely for x. Therefore F is 1–1 onto, and it is also $\mathbf{C}^\infty$ with everywhere nonvanishing Jacobian as required.■

2. Prove that

$$F(x,y,z) = \left(\frac{x}{2+y^2} + ye^z,\, \frac{x}{2+y^2} - ye^z,\, 2ye^z + z \right)$$

defines a $\mathbf{C}^\infty$ homeomorphism of $\mathbf{R}^3$ with a $\mathbf{C}^\infty$ inverse.

SOLUTION.

Use the same approach as in the previous problem. The map is $\mathbf{C}^\infty$ by construction, and its Jacobian is the following 3×3 determinant:

$$\begin{vmatrix} 1/(2+y^2) & -2\,xy/(2+y^2)^2 + e^z & ye^z \\ 1/(2+y^2) & -2\,xy/(2+y^2)^2 - e^z & -ye^z \\ 0 & 2e^z & 2ye^z + 1 \end{vmatrix}$$

One can use row and column operations to simplify the computation before writing everything out algebraically, but in any case the Jacobian is equal to

$$\frac{-2\,e^z}{2+y^2} \quad < \quad 0 \; .$$

The next step is to show that one can find unique solutions to the system of equations

$$(u, v, w) \quad = \quad \left(\frac{x}{2+y^2} + ye^z, \frac{x}{2+y^2} - ye^z, 2ye^z + z \right)$$

and here is a summary of how this can be done: Subtracting the second equation from the first yields $2\,y\,e^z = u - v$, and by the third equation the left hand side is equal to $w - z$. Therefore we can solve for z uniquely in terms of u, v and w. If we substitute this result into $2\,y\,e^z = u - v$ we also get a unique solution for y in terms of u, v and w. Finally, if we add the original first and second equations we obtain $u + v = 2\,x/(2+y^2)$. Since we already know that we can solve uniquely for y, this equation implies that we also get a unique solution for x terms of u, v and w.■

3. Let $f(x,y) = (x+y, x^2+y)$. Check that f meets the conditions to have a local inverse near $f(1,0) = (1,1)$, and if g is this local inverse find $Dg(1,1)$ without finding a formula for the inverse function explicitly.

SOLUTION.

One again begin by writing down the Jacobian:

$$\begin{vmatrix} 1 & 1 \\ 2\,x & 1 \end{vmatrix} \quad = \quad 1 - 2x$$

The right hand side is nonzero at $(1,0)$ and the derivative matrix $Df(1,0)$ is equal to

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \; .$$

By the Chain Rule, $Dg(1,1) = [Df(1,0)]^{-1}$, and by either Cramer's rule or one's favorite matrix inversion technique the latter is equal to

$$\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \; .■$$

4. Consider the mapping $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by $f(x,y) = (x^2+y^2, 2xy)$. Show that the Jacobian vanishes on the lines $y = \pm\,x$. What is the image of f? [*Hint:* Try using polar coordinates.]

The Inverse Function Theorem guarantees that f has a local inverse at $f(1,0) = (1,0)$. Find the inverse explicitly and describe a region on which it is defined.

SOLUTION.

Once again, compute the Jacobian:

$$\begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} \quad = \quad 2(x^2 - y^2)$$

This clearly vanishes on the lines $y = \pm x$. To find the image, use the hint to rewrite $f(x,y) = g[r,\theta] = (r^2, r^2 \sin 2\,\theta)$, where parentheses refer to rectangular coordinates and square brackets refer to polar coordinates. The polar expressions tell us that the image consists of all points (u,v) for which $u \geq 0$ and $|v| \leq u$.

To find the local inverse explicitly write $u = x^2 + y^2$ and $v = 2\,xy$. Solving this system of simultaneous quadratic equations is essentially an exercise in high school algebra. Since we are solving near $(1,0)$ we may divide by x more or less freely. It turns out that the solution for x and y in terms of u and v such that $(x,y) = (0,1)$ when $(u,v) = (0,1)$ is given by

$$x = \sqrt{\frac{u + \sqrt{u^2 - v^2}}{2}}, \quad y = \frac{v}{2\,x} \ .$$

We have not expressed x explicitly in terms of u and v in the second equation, but substitution of the first equation into the second enables one to write y explicitly in terms of u and v. This formula is valid for all (u,v) such that $u > 0$ and $|v| < u$.■

5. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by $f(v,w) = (v + w + e^{vw}, w + 2v + e^{-vw})$. Prove that there is a local inverse at $f(0,0) = (1,1)$ and if

$$g(x,y) = (\,v(x,y)\,,\ w(x,y)\,)$$

show that

$$\frac{\partial v}{\partial x}(1,1) + \frac{\partial w}{\partial x}(1,1) = \frac{\partial v}{\partial y}(1,1) \ .$$

SOLUTION.

In this problem $(x,y) = f(v,w)$ and g is supposed to be a local inverse to f at the indicated point. We need to check first that $Df(0,0)$ is invertible, so we evaluate it explicitly:

$$Df(v,w) = \begin{pmatrix} 1 + w\,e^{vw} & 1 + v\,e^{vw} \\ 2 - w\,e^{vw} & 1 - v\,e^{vw} \end{pmatrix} \qquad\qquad Df(0,0) = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

The determinant of the latter is nonzero, so the map is locally invertible and $Dg(1,1)$ is the inverse of $Df(0,0)$:

$$Dg(1,1) = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

Using this it is elementary to verify the equation in the exercise.■

V. Constructions on spaces

V.1 : Quotient spaces

Problem from Munkres, § 22, pp. 144 – 145

4. **(a)** Define an equivalence relation on the Euclidean coordinate plane by defining $(x_0, y_0) \sim (x_1, y_1)$ if and only if $x_0 + y_0^2 = x_1 + y_1^2$. It is homeomorphic to a familiar space. What is it? [*Hint:* Set $g(x, y) = x + y^2$.]

SOLUTION.

The hint describes a well-defined continuous map from the quotient space W to the real numbers. The equivalence classes are simply the curves $g(x, y) = C$ for various values of C, and they are parabolas that open to the left and whose axes of symmetry are the x-axis. It follows that there is a 1–1 onto continuous map from W to $\mathbf{R}$. How do we show it has a continuous inverse? The trick is to find a continuous map in the other direction. Specifically, this can be done by composing the inclusion of $\mathbf{R}$ in $\mathbf{R}^2$ as the x-axis with the quotient projection from $\mathbf{R}^2$ to W. This gives the set-theoretic inverse to $\mathbf{R}^2 \to W$ and by construction it is continuous. Therefore the quotient space is homeomorphic to $\mathbf{R}$ with the usual topology.■

(b) Answer the same question for the equivalence relation $(x_0, y_0) \sim (x_1, y_1)$ if and only if $x_0^2 + y_0^2 = x_1^2 + y_1^2$.

SOLUTION.

Here we define $g(x, y) = x^2 + y^2$ and the eqivalence classes are the circles $g(x, y) = C$ for $C > 0$ along with the origin. In this case we have a continuous 1–1 onto map from the quotient space V to the nonnegative real numbers, which we denote by $[0, \infty)$ as usual. To verify that this map is a homeomorphism, consider the map from $[0, \infty)$ to V given by composing the standard inclusion of the former as part of the x-axis with the quotient map $\mathbf{R}^2 \to V$. This is a set-theoretic inverse to the map from V to $[0, \infty)$ and by construction it is continuous.■

Additional exercises

0. Suppose that X is a space with the discrete topology and $\mathcal{R}$ is an equivalence relation on X. Prove that the quotient topology on $X/\mathcal{R}$ is discrete.

SOLUTION.

We claim that every subset of $X/\mathcal{R}$ is both open and closed. But a subset of the quotient is open and closed if and only if the inverse image has these properties, and every subset of a discrete space has these properties.■

1. If A is a subspace of X, a continuous map $r : X \to A$ is called a *retraction* if the restriction of r to A is the identity. Show that a retraction is a quotient map.

SOLUTION.

We need to show that $U \subset A$ is open if and only if $r^{-1}(U)$ is open in X. The ($\Longrightarrow$) implication is immediate from the continuity of r. To prove the other direction, note that $r \circ i = \mathrm{id}_A$ implies that

$$U \;=\; i^{-1}\left[r^{-1}(U)\right] \;=\; A \cap r^{-1}(U)$$

and thus if the inverse image of U is open in X then U must be open in A.■

2. Let $\mathcal{R}$ be an equivalence relation on a space X, and assume that $A \subset X$ contains points from every equivalence class of $\mathcal{R}$. Let $\mathcal{R}_0$ be the induced equivalence relation on A, and let

$$j : A/\mathcal{R}_0 \longrightarrow X/\mathcal{R}$$

be the associated $1-1$ correspondence of equivalence classes. Prove that j is a homeomorphism if there is a retraction $r : X \to A$ sending $\mathcal{R}$-equivalent points of X to $\mathcal{R}_0$-equivalent points of A. such that each set $r^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence class.

SOLUTION.

Let p_X and p_A denote the quotient space projections for X and A respectively. By construction, j is the unique function such that $j \circ p_A = p_X|A$ and therefore j is continuous. We shall define an explicit continuous inverse $k : X/\mathcal{R} \to A/\mathcal{R}_0$. To define the latter, consider the continuous map $p_A \circ r : X \to A/\mathcal{R}_0$. If $y\mathcal{R}z$ holds then $r(y)\mathcal{R}_0\, r(z)$, and therefore the images of y and z in $A/\mathcal{R}_0$ are equal. Therefore there is a unique continuous map k of quotient spaces such that $k \circ p_X = p_A \circ r$. This map is a set-theoretic inverse to j and therefore j is a homeomorphism.■

3. **(a)** Let 0 denote the origin in $\mathbf{R}^3$. In $\mathbf{R}^3 - \{0\}$ define $x\mathcal{R}y$ if y is a nonzero multiple of x (geometrically, if x and y lie on a line through the origin). Show that $\mathcal{R}$ is an equivalence relation; the quotient space is called the *real projective plane* and denoted by $\mathbf{RP}^2$.

SOLUTION.

The relation is reflexive because $x = 1 \cdot x$, and it is reflexive because $y = \alpha x$ for some $\alpha \neq 0$ implies $x = \alpha^{-1} y$. The relation is transitive because $y = \alpha x$ for $\alpha \neq 0$ and $z = \beta y$ for $y \neq 0$ implies $z = \beta\alpha x$, and $\beta\alpha \neq 0$ because the product of nonzero real numbers is nonzero.■

(b) Using the previous exercise show that $\mathbf{RP}^2$ can also be viewed as the quotient of S^2 modulo the equivalence relation $x \sim y \iff y = \pm x$. In particular, this shows that $\mathbf{RP}^2$ is compact. [*Hint:* Let r be the radial compression map that sends v to $|v|^{-1}v$.]

SOLUTION.

Use the hint to define r; we may apply the preceding exercise if we can show that for each $a \in S^2$ the set $r^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence class. By construction $r(v) = |v|^{-1}v$, so $r(x) = a$ if and only if x is a positive multiple of a (if $x = \rho\, a$ then $|x| = \rho$ and $r(x) = a$, while if $a = r(x)$ then by definition a and x are positive multiples of each other). Therefore if $x\mathcal{R}y$ then $r(x) = \pm\, r(y)$, so that $r(x)\mathcal{R}_0\, r(y)$ and the map

$$S^2/[x \equiv \pm x] \quad \longrightarrow \quad \mathbf{RP}^2$$

is a homeomorphism.■

4. In $D^2 = \{x \in \mathbf{R}^2 \mid |x| \leq 1\}$, consider the equivalence relation generated by the condition $x\mathcal{R}'y$ if $|x| = |y| = 1$ and $y = -x$. Show that this quotient space is homeomorphic to $\mathbf{RP}^2$.

[*Hints:* Use the description of $\mathbf{RP}^2$ as a quotient space of S^2 from the previous exercise, and let $h : D^2 \to S^2$ be defined by

$$h(x,y) \quad = \quad (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces. Why is $\overline{h}$ a $1-1$ and onto mapping? Finally, prove that $\mathbf{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.]

SOLUTION.

Needless to say we shall follow the hints in a step by step manner.

Let $h : D^2 \to S^2$ be defined by

$$h(x,y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces.

To show that $\overline{h}$ is well-defined it is only necessary to show that its values on the $\mathcal{R}'$-equivalence classes with two elements are the same for both representatives. If $\pi : S^2 \to \mathbf{RP}^2$ is the quotient projection, this means that we need $\pi \circ h(u) = \pi \circ h(v)$ if $|u| = |v| = 1$ and $u = -v$. This is immediate from the definition of the equivalence relation on S^2 and the fact that $h(w) = w$ if $|w| = 1$.

Why is $\overline{h}$ a $1-1$ and onto mapping?

By construction h maps the equivalence classe of points on the unit circle onto the points of S^2 with $z = 0$ in a 1–1 onto fashion. On the other hand, if u and v are distinct points that are not on the unit circle, then $h(u)$ cannot be equal to $\pm\, h(v)$. The inequality $h(u) \neq -h(v)$ follows because the first point has a positive z-coordinate while the second has a negative z-coordinate. The other inequality $h(u) \neq h(v)$ follows because the projections of these points onto the first two coordinates are u and v respectively. This shows that $\overline{h}$ is 1–1. To see that it is onto, recall that we already know this if the third coordinate is zero. But every point on S^2 with nonzero third coordinate is equivalent to one with positive third coordinate, and if $(x,y,z) \in S^2$ with $z > 0$ then simple algebra shows that the point is equal to $h(x,y)$.

Finally, prove that $\mathbf{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.

If the first statement is true, then the second one follows because the domain of $\overline{h}$ is a quotient space of a compact space and continuous maps from compact spaces to Hausdorff spaces are always closed. Since $\overline{h}$ is already known to be continuous, 1–1 and onto, this will prove that it is a homeomorphism.

So how do we prove that $\mathbf{RP}^2$ is Hausdorff? Let v and w be points of S^2 whose images in $\mathbf{RP}^2$ are distinct, and let P_v and P_w be their orthogonal complements in $\mathbf{R}^3$ (hence each is a 2-dimensional vector subspace and a closed subset). Since Euclidean spaces are Hausdorff, we can find an $\varepsilon > 0$ such that $N_\varepsilon(v) \cap P_v = \emptyset$, $N_\varepsilon(w) \cap P_w = \emptyset$, $N_\varepsilon(v) \cap N_\varepsilon(w) = \emptyset$, and $N_\varepsilon(-v) \cap N_\varepsilon(w) = \emptyset$. If T denotes multiplication by -1 on $\mathbf{R}^3$, then these conditions imply that the four open sets

$$N_\varepsilon(v), \quad N_\varepsilon(w), \quad N_\varepsilon(-v) = T\left(N_\varepsilon(v)\right), \quad N_\varepsilon(-w) = T\left(N_\varepsilon(w)\right)$$

are pairwise disjoint. This implies that the images of the distinct points $\pi(v)$ and $\pi(w)$ in $\mathbf{RP}^2$ lie in the disjoint subsets $\pi\left(N_\varepsilon(v)\right)$ and $\pi\left(N_\varepsilon(w)\right)$ respectively. These are open subsets in $\mathbf{RP}^2$ because their inverse images are given by the open sets $N_\varepsilon(v) \cup N_\varepsilon(-v)$ and $N_\varepsilon(w) \cup N_\varepsilon(-w)$ respectively.■

5. Suppose that X is a topological space with topology $\mathbf{T}$, and suppose also that Y and Z are sets with set-theoretic maps $f : X \to Y$ and $g : Y \to Z$. Prove that the quotient topologies satisfy the condition

$$(g \circ f)_* \mathbf{T} = g_* \left(f_* \mathbf{T}\right) .$$

(Informally, a quotient of a quotient is a quotient.)

SOLUTION.

`To be inserted.`

6. If Y is a topological space with a topology $\mathbf{T}$ and $f; X \to Y$ is a set-theoretic map, then the *induced topology* $f^*\mathbf{T}$ on X is defined to be the set of all subsets $W \subset X$ having the form $f^{-1}(U)$ for some open set $U \in \mathbf{T}$. Prove that $f^*\mathbf{T}$ defines a topology on X, that it is the unique smallest topology on X for which f is continuous, and that if $h : Z \to X$ is another set-theoretic map then

$$(f \circ h)^*\mathbf{T} = h^* \left(f^*\mathbf{T}\right) \ .$$

SOLUTION.

The object on the left hand side is the family of all sets having the form $(f \circ h)^{-1}(V)$ where V belongs to $\mathbf{T}$. As in the preceding exercise we have

$$(f \circ h)^{-1}(V) = h^{-1}\left(f^{-1}(V)\right)$$

so the family in question is just $h^* \left(f^*\mathbf{T}\right)$.■

7. Let X and Y be topological spaces, and define an equivalence relation $\mathcal{R}$ on $X \times Y$ by $(x, y) \sim (x', y')$ if and only if $x = x'$. Show that $X \times Y/\mathcal{R}$ is homeomorphic to X.

SOLUTION.

Let $p : X \times Y \to X$ be projection onto the first coordinate. Then $u\mathcal{R}v$ implies $p(u) = p(v)$ and therefore there is a unique continuous map $X \times Y/\mathcal{R} \to X$ sending the equivalence class of (x, y) to x. Set-theoretic considerations imply this map is 1–1 and onto, and it is a homeomorphism because p is an open mapping.■

8. Let $\mathcal{R}$ be an equivalence relation on a topological space X, let $\Gamma_{\mathcal{R}}$ be the graph of $\mathcal{R}$, and let $\pi : X \to X/\mathcal{R}$ be the quotient projection. Prove the following statements:

(a) If $X/\mathcal{R}$ satisfies the Hausdorff Separation Property then $\Gamma_{\mathcal{R}}$ is closed in $X \times X$.

SOLUTION.

`To be inserted.`

(b) If $\Gamma_{\mathcal{R}}$ is closed and π is open, then $X/\mathcal{R}$ is Hausdorff.

SOLUTION.

`To be inserted.`

(c) If $\Gamma_{\mathcal{R}}$ is open then $X/\mathcal{R}$ is discrete.

SOLUTION.

`To be inserted.`

V.2 : Sums and cutting and pasting

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

SOLUTION.

($\Longrightarrow$) If X is locally connected then so is every open subset. But each A_α is an open subset, so each is locally connected.■

($\Longleftarrow$) We need to show that for each $x \in X$ and each open set U containing x there is an open subset $V \subset U$ such that $x \in V$ and V is connected. There is a unique α such that $x = i_\alpha(a)$ for some $a \in A_\alpha$. Let $U_0 = i_\alpha^{-1}(U)$. Then by the local connectedness of A_α and the openness of U_0 there is an open connected set V_0 such that $x \in V_0 \subset U_0$. If $V = i_\alpha(V_0)$, then V has the required properties.■

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

SOLUTION.

`To be inserted.`

3. Prove that $\mathbf{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbf{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\mathrm{sign}(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x,\ \alpha(t)y,\ \frac{1}{2}t)$ where

$$\alpha(t) \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1, 1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1, 0, t)$ equivalent to $(1, 0, -t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

FURTHER DETAILS.

We shall fill in some of the reasons that were left unstated in the sketch given above.

Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T(A) = A$ and $T(B) = B$ [etc.]

This is true because if $T(v) = w$, then the third coordinates of both points have the same absolute values and of course they satisfy the same inequality relation with respect to $\frac{1}{2}$.

By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\mathrm{sign}(z) = \pm 1$,

This is true the third coordinates of all points in B are nonzero.

There is a homeomorphism h from $S^1 \times [-1,1]$ to A sending (x,y,t) to $(\alpha(t)x, \alpha(t)y, \frac{1}{2}t)$ where

$$\alpha(t)s \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

One needs to verify that h is 1–1 onto; this is essentially an exercise in algebra. Since we are dealing with compact Hausdorff spaces, continuous mappings that are 1–1 onto are automatically homeomorphisms.

This quotient space $[S^1 \times [-1,1]$ modulo the equivalence relation $u \sim v \iff u = \pm v]$ is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1,0,t)$ equivalent to $(1,0,-t)$, which yields the Möbius strip.

Let $\mathcal{A}$ and $\mathcal{B}$ be the respective equivalence relations on $S^1_+ \times [-1,1]$ and $S^1 \times [-1,1]$, and let **A** and **B** be the respective quotient spaces. By construction the inclusion $S^1_+ \times [-1,1] \subset S^1 \times [-1,1]$ passes to a continuous map of quotients, and it is necessary and sufficient to check that this map is 1–1 and onto. This is similar to a previous exercise. Points in $S^1 - S^1_+$ all have negative second coordinates and are equivalent to unique points with positive second coordinates. This implies that the mapping from **A** to **B** is 1–1 and onto at all points except perhaps those whose second coordinates are zero. For such points the equivalence relations given by $\mathcal{A}$ and $\mathcal{B}$ are identical, and therefore the mapping from **A** to **B** is also 1–1 and onto at all remaining points.∎

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by indentifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extenhds to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1,2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

SOLUTION.

`The first part will be posted later.` We can and shall view X as $A \cup_{\mathrm{id}} B$.
To construct the example where X is **not** homeomorphic to $A \cup_h B$, we follow the hint and try to find a homeomorphism of the four point space $\{\pm 1\} \times \{1,2\}$ to itself such that X is **not** homeomorphic to $A \cup_h B$ is connected; this suffices because we know that X is not connected. Sketches on paper or physical experimentation with wires or string are helpful in finding the right formula.

Specifically, the homeomorphism we want is given as follows:

$$\begin{array}{rcl} (-1,1) \in A_+ & \longrightarrow & (1,2) \in A_- \\ (1,1) \in A_+ & \longrightarrow & (1,1) \in A_- \\ (1,2) \in A_+ & \longrightarrow & (-1,1) \in A_- \\ (-1,2) \in A_+ & \longrightarrow & (-1,2) \in A_- \end{array}$$

The first of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{1\}$ lie in the same component of the quotient space, the second of these implies that the images of $S^1_- \times \{1\}$ and $S^1_+ \times \{1\}$ both lie in the same component, and the third of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{2\}$ also lie in the same component. Since the entire space is the union of the images of the connected subsets $S^1_\pm \times \{1\}$ and $S^1_\pm \times \{2\}$ it follows that $A \cup_h B$ is connected.∎

FOOTNOTE.

The argument in the first part of the exercise remains valid if A and B are open rather than closed subsets.■

5. *One-point unions.* One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X, x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point subset consisting of the base point is assumed to be closed.* If (Y, y) is another pointed space and $f : X \to Y$ is continuous, we shall say that f *is a base point preserving continuous map from* (X, x) *to* (Y, y) if $f(x) = y$, In this case we shall often write $f : (X, x) \to (Y, y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) \quad = \quad (X_1, x_1) \vee \cdots \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(a) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

SOLUTION.

For each j let $\mathbf{in}_j : X_j \to \coprod_k X_k$ be the standard injection into the disjoint union, and let

$$P : \coprod_k X_k \longrightarrow \bigvee_k (X_k, x_k)$$

be the quotient map defining the wedge. Define Y_j to be $P \circ \mathbf{in}_j(X_j)$. By construction the map $P \circ \mathbf{in}_j$ is continuous and 1–1; we claim it also sends closed subsets of X_j to closed subsets of the wedge. Suppose that $F \subset X_j$ is closed; then $P \circ \mathbf{in}_j(F)$ is closed in the wedge if and only if its inverse image under P is closed. But this inverse image is the union of the closed subsets $\mathbf{in}_j(F)$ and $\coprod_k \{x_k\}$ (which is a finite union of one point subsets that are assumed to be closed). It follows that Y_j is homeomorphic to $X + j$. The condition on $Y_k \cap Y_\ell$ for $k \neq \ell$ is an immediate consequence of the construction.

The assertion that the wedge is Hausdorff if and only if each summand is follows because a subspace of a Hausdorff space is Hausdorff, and a finite union of closed Hausdorff subspaces is always Hausdorff (by a previous exercise).

To verify the assertions about compactness, note first that for each j there is a continuous collapsing map q_j from $\vee_k (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is compact, then its continuous under q_j, which is the image of X_j, must also be compact. Conversely if the sets X_j are compact for all j, then the (finite!) union of their images, which is the entire wedge, must be compact.

To verify the assertions about connectedness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\ (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is connected, then its continuous under q_j, which is the image of X_j, must also be connected. Conversely if the sets X_j are connected for all j, then the union of their images, which is the entire wedge, must be connected because all these images contain the base point. Similar statements hold for arcwise connectedness and follow by inserting "arcwise" in front of "connected" at every step of the argument.■

(b) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k\ X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

SOLUTION.

To prove existence, first observe that there is a unique continuous map $\widetilde{F} : \coprod_k X_k \to Y$ such that $\mathbf{in}_j \circ \widetilde{F} = F_j$ for all j. This passes to a unique continuous map F on the quotient space $\vee_k\ (X_k, x_k)$ because $\widetilde{F}$ is constant on the equivalence classes associated to the quotient projection P. This constructs the map we want; uniqueness follows because the conditions prescribe the definition at every point of the wedge.■

(c) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j(F)$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

SOLUTION.

Strictly speaking, one should verify that the so-called weak topology is indeed a topology on the wedge. We shall leave this to the reader.

To prove [1], note that ($\Longrightarrow$) is trivial. For the reverse direction, we need to show that if E is closed in Y then $h^{-1}(E)$ is closed with respect to the so-called weak topology we have defined. The subset in question is closed with respect to this topology if and only if $h^{-1}(E) \cap \varphi(X_j)$ is closed in $\varphi(X_j)$ for all j, and since φ_j maps its domain homeomorphically onto its image, the latter is true if and only if $\varphi^{-1} \circ h^{-1}(E)$ is closed in X_j for all j. But these conditions hold because each of the maps $\varphi_j \circ h$ is continuous. To prove [2], note first that there is a unique set-theoretic map, and then use [1] to conclude that it is continuous.■

(d) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

SOLUTION.

For each j let $y_j \in X_j$ be a point other than x_j, and consider the set E of all points y_j. This is a closed subset of the wedge because its intersection with each set $\varphi(X_j)$ is a one point subset and hence closed. In fact, every subset of E is also closed by a similar argument (the intersections with the summands are either empty or contain only one point), so E is a discrete closed subset of the wedge. Compact spaces do not have infinite discrete closed subspaces, and therefore it follows that the infinite wedge with the weak topology is not compact.■

Remark. If each of the summands in **(d)** is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j) is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbf{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 \quad + \ + y^2 \quad = \quad \frac{1}{2^{2k}} \ .$$

As usual, drawing a picture may be helpful. The k^{th} circle has center $(1/2^k,\ 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

SKETCHES OF VERIFICATIONS OF ASSERTIONS.

If we are given an infinite sequence of compact Hausdorff pointed spaces $\{\,(X_n, x_n)\,\}$ we can put a compact Hausdorff topology on their wedge as follows. Let W_k be the wedge of the first k spaces; then for each k there is a continuous map

$$q_k : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad W_k$$

(with the so-called weak topology on the wedge) that is the identity on the first k summands and collapses the remaining ones to the base point. These maps are in turn define a continuous function

$$\mathbf{q} : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad \prod_k W_k$$

whose projection onto W_k is q_k. This mapping is continuous and 1–1; if its image is closed in the (compact!) product topology, then this defines a compact Hausdorff topology on the infinite wedge $\vee_n (X_n, x_n)$.

Here is one way of verifying that the image is closed. For each k let $c_k : W_k \to W_{k-1}$ be the map that is the identity on the first $(k-1)$ summands and collapses the last one to a point. Then we may define a continuous map C on $\prod_{k \geq 1} W_k$ by first projecting onto the product $\prod_{k \geq 2} W_k$ (forget the first factor) and then forming the map $\prod_{k \geq 2} W_k$. The image of $\mathbf{q}$ turns out to be the set of all points $\mathbf{x}$ in the product such that $C(\mathbf{x}) = \mathbf{x}$. Since the product is Hausdorff the image set is closed in the product and thus compact.

A comment about the compactness of the Hawaiian earring E might be useful. Let F_k be the union of the circles of radius 2^{-j} that are contained in E, where $j \leq k$, together with the closed disk bounded by the circle of radius $2^{-(k+1)}$ in E. Then F_k is certainly closed and compact. Since E is the intersection of all the sets F_k it follows that E is also closed and compact.■