SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 5

Fall 2003

VI. Spaces with additional properties

VI.1 : Second countable spaces

Problems from Munkres, § 30, *pp.* 194 – 195

9. [*First part only*] *Let X be a Lindelöf space, and suppose that A is a closed subset of X. Prove that A is Lindelöf.*

SOLUTION.

The statement and proof are parallel to a result about compact spaces in the course notes, the only change being that "compact" is replaced by "Lindelöf."∎

10. Show that if X is a countable product of spaces having countable dense subsets, then X also has a countable dense subset.

SOLUTION.

Suppose first that X is a finite product of spaces Y_i such that each Y_i has a countable dense subset D_i. Then $\prod_i D_i$ is countable and

$$X \quad = \quad \prod_i Y_i \quad = \quad \prod_i \overline{D_i} \quad = \quad \overline{\prod_\alpha D_i} \ .$$

Suppose now that X is countably infinite. The same formula holds, but the product of the D_i's is not necessarily countable. To adjust for this, pick some point $\delta_j \in D_j$ for each j and consider the set E of all points $(a_0, a_1, \cdots)$ in $\prod_j D_j$ such that $a_j = \delta_j$ for all but at most finitely many values of j. This set is countable, and we claim it is dense. It suffices to show that every basic open subset contains at least one point of E. But suppose we are given such a set $V = \prod_j V_j$ where V_j is open in X_j and $V_j = X_j$ for all but finitely many j; for the sake of definiteness, suppose this happens for $j > M$. For $j \leq M$, let $b_j \in D_j \cap V_j$; such a point can be found since D_j is dense in X_j. Set $b_j = \delta_j$ for $j > M$. If we let $b = (b_0, b_1, \cdots)$, then it then follows that $b \in E \cap V$, and this implies E is dense in the product.∎

13. Show that if X has a countable dense subset, then every collection of disjoint open subsets in X is countable.

SOLUTION.

This is similar to the proof that an open subset of $\mathbf{R}^n$ has only countably many components.∎

14. Show that if X is compact and Y is Lindelöf, then $X \times Y$ is Lindelöf.

SOLUTION.

This is essentially the same argument as the one showing that a product of two compact spaces is compact. The only difference is that after one constructs an open covering of Y at one step in the proof, then one only has a countable subcovering and this leads to the existence of a countable subcovering of the product. Details are left to the reader.■

Additional exercises

1. If $(X, \mathbf{T})$ is a second countable Hausdorff space, prove that the cardinalities of both X and $\mathbf{T}$ are less than or equal to $2^{\aleph_0}$. (Using the formulas for cardinal numbers in Section I.3 of the course notes and the separability of X one can prove a similar inequality for $\mathbf{BC}(X)$.)

SOLUTION.

`To be inserted.`

2. *Separability and subspaces.* The following example shows that a closed subspace of a separable Hausdorff space is not necessarily separable.

(a) Let X be the upper half plane $\mathbf{R} \times [0, \infty)$ and take the topology generated by the usual metric topology plus the following sets:

$$T_\varepsilon(x) \quad = \quad \{\,(x,0)\,\} \;\cup\; N_\varepsilon\,(\,(x,\varepsilon)\,)\,, \quad \text{where} \;\; x \in \mathbf{R} \;\; \text{and} \;\; \varepsilon \;>\; 0\}$$

Geometrically, one takes the interior region of the circle in the upper half plane that is tangent to the x-axis at $(x, 0)$ and adds the point of tangency. — Show that the x-axis is a closed subset and has the discrete topology.

SOLUTION.

The x-axis is closed because it is closed in the ordinary Euclidean topology and the "new" topology contains the Euclidean topology; therefore the x-axis is closed in the "new" topology. The subspace topology on the x-axis is the discrete topology intersection of the open set $T_\varepsilon(x)$ with the real axis is $\{\,x\,\}$.■

(b) Explain why the space in question is Hausdorff. [*Hint:* The topology contains the metric topology. If a topological space is Hausdorff and we take a larger topology, why is the new topology Hausdorff?]

SOLUTION.

In general if $(X, \mathbf{T})$ is Hausdorff and $\mathbf{T} \subset \mathbf{T}^*$ then $(X, \mathbf{T}^*)$ is also Hausdorff, for a pair of disjoint $\mathbf{T}$-open subsets containing distinct point u and v will also be a pair of $\mathbf{T}$-open subsets with the same properties.■

(c) Show that the set of points (u, v) in X with $v > 0$ and $u, v \in \mathbf{Q}$ is dense. [*Hint:* Reduce this to showing that one can find such a point in every set of the form $T_\varepsilon(x)$.]

SOLUTION.

To show that a subset D of a topological space X is dense, it suffices to show that the intersection of D with every nonempty open subset in some base $\mathcal{B}$ is nonempty (why?). Thus we need to show that the generating set is a base for the topology and that every basic open subset contains a point with rational coordinates.

The crucial point to showing that we have a base for the topology is to check that if U and V are basic open subsets containing some point $z = (x, y)$ then there is some open subset in the generating collection that contains z and is contained in $U \cap V$. If U and V are both basic open subsets we already know this, while if $z \in T_\varepsilon(a) \cap T_\delta(b)$ there are two cases depending upon whether or not z lies on the x-axis; denote the latter by A. If $z \notin A$ then it lies in the metrically open subsets $T_\varepsilon(a) - A$ and $T_\varepsilon(b) - A$, and one can find a metrically open subset that contains z and is contained in the intersection. On the other hand, if $z \in A$, then $T_\varepsilon(a) \cap A = \{a\}$ and $T_\delta(b) \cap A = \{b\}$ imply $a = b$, and the condition on intersections is immediate because the intersection of the subsets is either $T_\varepsilon(a)$ or $T_\delta(a)$ depending upon whether $\delta \leq \varepsilon$ or vice versa.■

3. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

SOLUTION.

For each property $\mathcal{P}$ given in the exercise, the space X has property $\mathcal{P}$ if and only if each A_α does and there are only finitely many α for which A_α is nonempty. The verifications for the separate cases are different and will be given in reverse sequence.

The Lindelöf property.

The proof in this case is the same as the proof we gave for compactness in an earlier exercises with "countable" replacing "finite" throughout.■

Separability.

($\Longrightarrow$) Let D be the countable dense subset. Each A_α must contain some point of D, and by construction this point is not contained in any of the remaining sets A_β. Thus we have a 1–1 function from $\mathcal{A}$ to D sending α to a point $d(\alpha) \in A_\alpha \cap D$. This implies that the cardinality of $\mathcal{A}$ is at most $|D| \leq \aleph_0$.■

($\Longleftarrow$) If D_α is a dense subset of A_α and $\mathcal{A}$ is countable, then $\cup_\alpha\, D_\alpha$ is a countable dense subset of $\mathcal{A}$.■

Second countability.

($\Longrightarrow$) Since a subspace of a second countable space is second countable, each A_α must be second countable. Since the latter condition implies both separability and the Lindelöf property, the preceding arguments show that only countably many summands can be nontrivial.

($\Longleftarrow$) If $\mathcal{A}$ is countable and $\mathcal{B}_\alpha$ is a countable base for A_α then $\cup_\alpha\, \mathcal{B}_\alpha$ determines a countable base for X (work out the details!).■

VI.2 : Compact spaces – II

Problems from Munkres, § 28, pp. 181 – 182

6. Prove that if $(X, \mathbf{d})$ is a metric space and $f : X \to X$ is a distance preserving map (an *isometry*), then f is $1-1$ onto and hence a homeomorphism. [*Hint:* If $a \notin f(X)$ choose $\varepsilon > 0$ so that the ε-neighborhood of a is disjoint from $f(X)$. Set $x_0 = a$ and $x_{n+1} = f(x_n)$ in general. Show that $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$.]

SOLUTION.

Follow the hint to define a and the sequence. The existence of ε is guaranteed because $a \notin f(X)$ and the compactness of $f(X)$ imply that the continuous function $g(x) = \mathbf{d}(a, f(x))$ is positive valued, and it is bounded away from 0 because it attains a minimum value. Since $x_k \in f(X)$ for all k it follows that $\mathbf{d}(a, x_k) > \varepsilon$ for all k. Given $n \neq m$ write $m = n + k$; reversing the roles of m and n if necessary we can assume that $k > 0$. If f is distance preserving, then so is every n-fold iterated composite $\circ^n f$ of f with itself. Therefore we have that

$$\mathbf{d}(x_n, x_m) \quad = \quad \mathbf{d}\left(\circ^n f(a), \circ^n f(x_k)\right) \quad = \quad \mathbf{d}(a, x_k) \quad > \quad \varepsilon$$

for all distinct nonnegative integers m and n. But this cannot happen if X is compact, because the latter implies that $\{ x_n \}$ has a convergent subsequence. This contradiction implies that our original assumption about the existence of a point $a \notin f(X)$ is false, so if is onto. On the other hand $x \neq y$ implies

$$0 \quad < \quad \mathbf{d}(x, y) \quad = \quad \mathbf{d}\left(f(x), f(y) \right)$$

and thus that f is also 1–1. Previous results now also imply that f is a homeomorphism onto its image.∎

FOOTNOTE.

To see the need for compactness rather than (say) completeness, consider the map $f(x) = x+1$ on the set $[0, +\infty)$ of nonnegative real numbers.∎

Additional exercises

1. Let X be a compact Hausdorff space, let Y be a Hausdorff space, and let $f : X \to Y$ be a continuous map such that f is locally 1–1 (each point x has a neighborhood U_x such that $f|U_x$ is 1–1) and there is a closed subset $A \subset X$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is 1–1. [*Hint:* A map g is 1–1 on a subset B is and only if

$$B \times B \cap (g \times g)^{-1}(\Delta_Y) \quad = \quad \Delta_B$$

where Δ_S denotes the diagonal in $S \times S$. In the setting of the exercise show that

$$(f \times f)^{-1}(\Delta_Y) \quad = \quad \Delta_X \cup D',$$

where D' is closed and disjoint from the diagonal. Also show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.]

SOLUTION.

Follow the steps in the hint.

> *A map g is $1-1$ on a subset B is and only if $B \times B \cap (g \times g)^{-1}(\Delta_Y) = \Delta_B$ where Δ_S denotes the diagonal in $S \times S$.*

This is true because the set on the left hand side of the set-theoretic equation is the set of all (b, b') such that $g(b) = g(b')$. If there is a nondiagonal point in this set then the function is not 1–1, and conversely if the function is 1–1 then there cannot be any off-diagonal terms in the set.∎

> *Show that if $D' = (f \times f)^{-1}(\Delta_Y) - \Delta_X$, then D' is closed and disjoint from the diagonal.*

Since f is locally 1–1, for each $x \in X$ there is an open set U_x such that $f|U_x$ is 1–1; by the first step we have

$$U_x \times U_x \cap (f \times f)^{-1}(\Delta_Y) \quad = \quad \Delta_{U_x} \ .$$

If we set $W = \cap_x\, U_x \times U_x$ then W is an open neighborhood of Δ_X and by construction we have

$$(f \times f)^{-1}(\Delta_Y) \cap W \quad = \quad \Delta_X$$

which shows that Δ_X is open in $(f \times f)^{-1}(\Delta_Y)$ and thus its relative complement in the latter — which is D' — must be closed.■

Show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.

Since $f|A$ is 1–1 we have $A \times A \cap (f \times f)^{-1}(\Delta_Y) = \Delta_A$, which lies in $\Delta_X = (f \times f)^{-1}(\Delta_Y) - D'$. Since A is a compact subset of the open set $X \times X - D'$, by Wallace's Theorem there is an open set U such that

$$A \times A \quad \subset \quad U \times U \quad \subset \quad X \times X - D' \ .$$

The first part of the proof now implies that $f|U$ is 1–1.■

2. Let U be open in $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^n$ be a $\mathbf{C}^1$ map such that $Df(x)$ is invertible for all $x \in U$ and there is a compact subset $A \subset U$ such that $f|A$ is $1-1$. Prove that there is an open neighborhood V of A such that $f|V$ is a homeomorphism onto its image.

SOLUTION.

tt To be inserted.

3. Let $\mathbf{d}_p$ be the metric on the integers constructed in Exercise I.1.1, and let $\widehat{\mathbf{Z}_p}$ be the completion of this metric space. Prove that $\widehat{\mathbf{Z}_p}$ is (sequentially) compact. [*Hint:* For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers. Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a \bmod p^{r+1}$. Now let $\{\, a_n \,\}$ be an infinite sequence of integers, and assume the sequence takes infinitely many distinct values (otherwise the sequence obviously has a convergent subsequence). Find a sequence of positive integers $\{\, b_r \,\}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many points in the sequence and $b_{r+1} \equiv b_r \bmod p^{r+1}$. Form a subsequence of $\{\, a_n \,\}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

SOLUTION.

As usual, we follow the steps in the hint.

For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers.

If n is an integer, use the long division property to write $n = p^{r+1}a$ where $0 \leq a < p^{r+1}$. We then have $\mathbf{d}_p(n, a) \leq p^{-(r+1)} < p^r$.■

Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a \bmod p^{r+1}$.

Suppose we do long division by p^{r+2} rather than p^{r+1} and get a new remainder a'. How is it related to a? Very simply, a' must be one of the p nonnegative integers b such that $0 \leq b < p^{r+2}$ and $b \equiv a \bmod p^{r+1}$. Since $\mathbf{d}(u, v) < p^{-r}$ if and only if $\mathbf{d}_p(u, v,) \leq p^{-(r+1)}$, it follows that $N_{p^{-r}}(a)$ is a union of p open subsets of the form $N_{p^{-(r+1)}}(b)$ as claimed (with the numbers b satisfying the asserted conditions).

Find a sequence of nonnegative integers $\{ b_r \}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many terms in the sequence and $b_{r+1} \equiv b_r \bmod p^{r+1}$.

This is very similar to a standard proof of the Bolzano-Weierstrass Theorem in real variables. Infinitely many terms of the original sequence must lie in one of the sets $N_1(b_0)$ where $1 \leq b < p$. Using the second step we know there is some b_1 such that $b_1 \equiv b_0 \bmod p$ and infinitely many terms of the sequence lie in $N_{p^{-1}}(b_1)$, and one can continue by induction (fill in the details!).■

Form a subsequence of $\{ a_n \}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

We know that the neighborhoods in question contain infinitely many values of the sequence, and this allows us to find $n(k)$ recursively. It remains to show that the construction yields a Cauchy sequence. The key to this is to observe that $a_{n(k)} \equiv b_k \bmod p^{k+1}$ and thus we also have

$$\mathbf{d}_p \left(a_{n(k+1)}, a_{n(k)} \right) \quad < \quad \frac{1}{p^k} \ .$$

Similarly, if $\ell > k$ then we have

$$\mathbf{d}_p \left(a_{n(\ell)}, a_{n(k)} \right) \quad < \quad \frac{1}{p^{\ell-1}} \ + \ \cdots \ + \ \frac{1}{p^k} \ .$$

Since the geometric seris $\sum_k p^{-k}$ converges, for every $\varepsilon > 0$ there is an M such that $\ell, k \geq M$ implies the right had side of the displayed inequality is less than ε, and therefore it follows that the constructed subsequence is indeed a Cauchy sequence. By completeness (of the completion) this sequence converges. Therefore $\widehat{\mathbf{Z}_p}$ is (sequentially) compact.■

VI.3 : Separation axioms

Problem from Munkres, § 26*, pp.* 170 – 172

11. Let X be a compact Hausdorff space, and let $\{ A_\alpha \}$ be a family of nonempty closed connected subsets ordered by inclusion. Prove that $Y = \cap_\alpha A_\alpha$ is connected. [*Hint:* If $C \cup D$ is a separation of Y, choose disjoint open sets U and V of X containing C and D respectively and show that

$$B \quad = \quad \bigcap_\alpha (A_\alpha - (U \cup V))$$

is nonempty.]

SOLUTION.

Follow the suggestion of the hint to define C, D and to find U, V. There are disjoint open subset sets U, $V \subset X$ containing C and D respectively because a compact Hausdorff space is $\mathbf{T_4}$.

For each α the set $B_\alpha = A_\alpha - (U \cup V)$ is a closed and hence compact subset of X. If each of these subsets is nonempty, then the linear ordering condition implies that the family of closed compact subsets B_α has the finite intersection property; specifically, the intersection

$$B_{\alpha(1)} \cap \ \ldots \ \cap B_{\alpha(k)}$$

is equal to $B_{\alpha(j)}$ where $A_{\alpha(j)}$ is the smallest subset in the linearly ordered collection

$$\{A_{\alpha(1)}, \ \dots \ , A_{\alpha(k)}\} \ .$$

Therefore by compactness it will follow that the intersection

$$\bigcap_\alpha B_\alpha \quad = \quad \left(\bigcap_\alpha A_\alpha \right) - (U \cup V)$$

is nonempty. But this contradicts the conditions $\cap_\alpha A_\alpha = C \cup D \subset U \cup V$. Therefore it follows that the $\cap_\alpha A_\alpha$ must be connected.

Therefore we only need to answer the following question: Why should the sets $A_\alpha - (U \cup V)$ be nonempty? If the intersection is empty then $A_\alpha \subset U \cup V$. By construction we have $C \subset A_\alpha \cap U$ and $D \subset A_\alpha \cap V$, and therefore we can write A_α as a union of two nonempty disjoint open subsets. However, this contradicts our assumption that A_α is connected and therefore we must have $A_\alpha - (U \cup V) \neq \emptyset$.■

Problems from Munkres, § 33, *pp.* 212 – 214

2. **(a)** [*For metric spaces.*] Show that a connected metric space having more than one point is uncountable.

SOLUTION.

6. A space X is said to be *perfectly normal* if every closed set in X is a $\mathbf{G}_\delta$ subset of X. [*Note:* The latter are defined in Exercise 1 on page 194 of Munkres and the reason for the terminology is also discussed at the same time.]

(a) Show that every metric space is perfectly normal.

SOLUTION.

Let $(X, \mathbf{d})$ be a metric space, let $A \subset X$ be closed and let $f(x) = \mathbf{d}(x, A)$. Then

$$A \quad = \quad f^{-1}(\{0\}) \quad = \quad \bigcap_n f^{-1}\Big(\, (-\tfrac{1}{n}, \tfrac{1}{n}) \, \Big)$$

presents A as a countable intersection of open subsets.■

8. Let X be a completely regular space, and let A and B be disjoint closed subsets such that A is compact. Prove that there is a continuous function $f : X \to [0,1]$ that is 0 on A and 1 on B.

SOLUTION.

For each $a \in A$ there is a continuous function $f_a : X \to [-1,1]$ that is -1 at a and 0 on B. Let $U_a = f_a^{-1}([-1,0)\,)$. Then the sets U_a define an open covering of A and hence there is a finite subcovering corresponding to $U_{a(1)}, \ \cdots \ , U_{a(k)}$. Let f_i be the function associated to $a(i)$, let g_i be the maximum of f_i and 0 (so g_i is continuous by a previous exercise), and define

$$f = \prod_{i=1}^{k} g_k \ .$$

By construction the value of f is 1 on B because each factor is 1 on B, and $f = 0$ on $\cup_i U_{a(i)}$ because $g_j = 0$ on $U_{a(j)}$; since the union contains A, it follows that $f = 0$ on A.■

Additional exercises

1. If $(X, \mathbf{T})$ is compact Hausdorff and $\mathbf{T}^*$ is strictly contained in $\mathbf{T}$, prove that $(X, \mathbf{T}^*)$ is compact but not Hausdorff.

SOLUTION.

`To be inserted.`

2. **(a)** Prove that a topological space is $\mathbf{T_3}$ if and only if it is $\mathbf{T_1}$ and there is a basis $\mathcal{B}$ such that for every $x \in X$ and every open set $V \in \mathcal{B}$ containing x, there is an open subset $W \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset V$.

SOLUTION.

$(\Longrightarrow)$ Suppose that X is $\mathbf{T_3}$, let $x \in X$ and let V be a basic open subset containing x. Then there is an open set U in X such that $x \in U \subset \overline{U} \subset V$. Sinc $\mathcal{B}$ is a basis for the topology, one can find a basic open subset W in $\mathcal{B}$ such that $x \in W \subset U$, and thus we have $x \in W \subset \overline{W} \subset \overline{U} \subset V$.■
$(\Longleftarrow)$ Suppose that X is $\mathbf{T_1}$ and satisfies the condition in the exercise. If U is an open subset and $x \in U$, let V be a basic open set from $\mathcal{B}$ such that $x \in V \subset U$, and let W be the basic open set that exists by the hypothesis in the exercise. Then we have $x \in W \subset \overline{W} \subset V \subset U$ and therefore X is regular.■

(b) Prove that the space constructed in Exercise VI.1.2 is $\mathbf{T_3}$. [*Hint:* Remember that the "new" topology contains the usual metric topology.]

SOLUTION.

By the first part of the exercise we only need to show this for points in basic open subsets. If the basic open subset comes from the metric topology, this follows because the metric topology is $\mathbf{T_3}$; note that the closure in the new topology might be smaller than the closure in the metric topology, but if a metrically open set contains the metric closure it also contains the "new" closure. If the basic open subset has the form $T_\varepsilon(a)$ for some a and z belongs to this set, there are two cases depending upon whether or not z lies on the x-axis, which we again call A. If $z \notin A$, then z lies in the metrically open subset $T_\varepsilon(a) - A$, and one gets a subneighborhood whose closure lies in the latter exactly as before. On the other hand, if $z \in A$ then $z = (a, 0)$ and the closure of the set $T_\delta(a)$ in either topology is contained in the set $T_\varepsilon(a)$.■

3. If X is a topological space and $A \subset X$ is nonempty then X/A (in words, "X mod A" or "X modulo A collapsed to a point") is the quotient space whose equivalence classes are A and all one point subsets $\{x\}$ such that $x \notin A$. Geometrically, one is collapsing A to a single point.

(a) Suppose that A is closed in X. Prove that X/A is Hausdorff if either X is compact Hausdorff or X is metric (in fact, if X is $\mathbf{T_3}$).

SOLUTION.

`To be inseerted.`

(b) Still assuming A is closed but not making any assumptions on X (except that it be nonempty), show that the quotient map $X \to X/A$ is always closed but not necessarily open. [*Note:* For reasons that we shall not discuss, it is appropriate to define $X/\emptyset$ to be the disjoint union $X \sqcup \{\emptyset\}$.]

SOLUTION.

`To be inseerted.`

VI.4 : Local compactness and compactifications

Problems from Munkres, § 38, pp. 241 – 242

2. Show that the bounded continuous function $g : (0,1) \to [-1,1]$ defined by $g(x) = \cos(1/x)$ cannot be extended continuously to the compactification in Example 3 [on page 238 of Munkres]. Define an embedding $h : (0,1) \to [-1,1]^3$ such that the functions x, $\sin(1/x)$ and $\cos(1/x)$ are all extendible to the compactifcation given by the closure of the image of h.

SOLUTION.

We need to begin by describing the compactification mentioned in the exercise. It is given by taking the closure of the embedding (homeomorphism onto its image) $g : (0,1) \to [-1,1]^2$ that is inclusion on the first coordinate and $\sin(1/x)$ on the second.

Why is it impossible to extend $\cos(1/x)$ to the closure of the image? Look at the points in the image with coordinates $(1/k\pi, \sin k\pi)$ where k is a positive integer. The second coordinates of these points are always 0, so this sequence converges to the origin. If there is a continuous extension F, it will follow that

$$F(0,0) \quad = \quad \lim_{k\to\infty} \cos\left(\frac{1}{1/k\pi}\right) \quad = \quad \cos k\pi \ .$$

But the terms on the right hand side are equal to $(-1)^k$ and therefore do not have a limit as $k \to \infty$. Therefore no continuous extension to the compactification exists.

One can construct a compactification on which x, $\sin(1/x)$ and $\cos(1/x)$ extend by taking the closure of the image of the embedding $h : (0,1) \to [-1,1]^3$ defined by

$$h(x) \quad = \quad (x, \sin(1/x), \cos(1/x)\,) \ .$$

The continuous extensions are given by restricting the projections onto the first, second and third coordinates.■

3. [*Just give a necessary condition on the topology of the space.*] Under what conditions does a metrizable space have a metrizable compactification?

SOLUTION.

If A is a dense subset of a compact metric space, then A must be second countable because a compact metric space is second countable and a subspace of a second countable space is also second countable.

This condition is also sufficient, but the sufficiency part was not assigned because it requires the Urysohn Metrization Theorem. The latter says that a $\mathbf{T_3}$ and second countable topological space is homeomorphic to a product of a countably infinite product of copies of $[0,1]$; this space is compact by Tychonoff's Theorem, and another basic result not covered in the course states that a countable product of metrizable spaces is metrizable in the product topology (see Munkres, Exercise 3 on pages 133–134). So if X is metrizable and second countable, the Urysohn Theorem maps it homeomorphically to a subspace of a compact metrizable space, and the closure of its image will be a metrizable compactification of X.■

Definition. If $f : X \to Y$ is continuous, then f is *proper* (or *perfect*) if for each compact subset $K \subset Y$ the inverse image $f^{-1}(K)$ is a compact subset of X.

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact T_2 spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet|X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

SOLUTION.

$(\Longrightarrow)$ Suppose that f is proper and U is open in $Y^\bullet$. There are two cases depending upon whether $\infty_Y \in U$. If not, then $U \subset Y$ and thus $[f^\bullet]^{-1}(U) = f^{-1}(U)$ is an open subset of X; since X is open in $X^\bullet$ it follows that $f^{-1}(U)$ is open in $X^\bullet$. On the other hand, if $\infty_Y \in U$ then $Y - U$ is compact, and since f is proper it follows that

$$C \;=\; f^{-1}(Y-U) \;=\; X - f^{-1}\left(U - \{\infty_Y\}\right)$$

is a compact, hence closed, subset of X and $X^\bullet$. Therefore

$$[f^\bullet]^{-1}(U) \;=\; f^{-1}(U) \cup \{\infty_X\} \;=\; X^\bullet - C$$

is an open subset of $X^\bullet$.■

$(\Longleftarrow)$ Suppose that $f^\bullet$ is continuous and $A \subset Y$ is compact. Then

$$f^{-1}(A) = [f^\bullet]^{-1}(A)$$

is a closed, hence compact subset of $X^\bullet$ and likewise it is a compact subset of X.■

2. Prove that a proper map of noncompact locally compact Hausdorff spaces is closed.

SOLUTION.

Let $f : X \to Y$ be a proper map of noncompact locally compact Hausdorff spaces, and let $f^\bullet$ be its continuous extension to a map of one point compactifications. Since the latter are compact Hausdorff it follows that $f^\bullet$ is closed. Suppose now that $F \subset X$ is closed. If F is compact, then so is $f(F)$ and hence the latter is closed in Y. Suppose now that F is not compact, and consider the closure E of F in $X^\bullet$. This set is either F itself or $F \cup \{\infty_X\}$ (since F is its own closure in X it follows that $E \cap X = F$). Since the closed subset $E \subset X^\bullet$ is compact, clearly $E \neq F$, so this implies the second alternative. Once again we can use the fact that $f^\bullet$ is closed to show that $f^\bullet(E) = f(F) \cup \{\infty_Y\}$ is closed in $Y^\bullet$. But the latter equation implies that $f(F) = f^\bullet(F) \cap Y$ is closed in Y.■

3. If $\mathbf{F}$ is the reals or complex numbers, prove that every polynomial map $p : \mathbf{F} \to \mathbf{F}$ is proper. [Hint: Show that

$$\lim_{|z|\to\infty} \big|p(z)\big| = \infty$$

and use the characterization of compact subsets as closed and bounded subsets of $\mathbf{F}$.]

SOLUTION.

Write the polynomial as

$$p(z) \;=\; \sum_{k=0}^{n} a_k\, z^k$$

where $a_n \neq 0$ and $n > 0$, and rewrite it in the following form:

$$a_n z^n \cdot \left(1 \ + \ \sum_{k=1}^{n-1} \frac{a_k}{a_n} \frac{1}{z^{n-k}} \right)$$

The expression inside the parentheses goes to 1 as $n \to \infty$, so we can find $N_0 > 0$ such that $|z| \geq N_0$ implies that the abolute value (or modulus) of this expression is at least $\frac{1}{2}$.

Let $M > 0$ be arbitrary, and define

$$N_1 \ = \ \left(\frac{2M_0}{|a|_n} \right)^{1/n} .$$

Then $|z| > \max(N_0 . N_1)$ implies $|p(z)| > M$. This proves the limit formula.

To see that p is proper, suppose that K is a compact subset of $\mathbf{F}$, and choose $M > 0$ such that $w \in K$ implies $|w| \leq M$. Let N be the maximum of N_1 and N_2, where these are defined as in the preceding paragraph. We then have that the closed set $p^{-1}(K)$ lies in the bounded set of points satisfying $|z| \leq N$, and therefore $p^{-1}(K)$ is a compact subset of $\mathbf{F}$.■

4. Let ℓ^2 be the complete metric space described above, and view $\mathbf{R}^n$ as the subspace of all sequences with $x_k = 0$ for $k > n$. Let $A_n \subset \ell^2 \times \mathbf{R}$ be the set of all ordered pairs (x, t) with $x \in \mathbf{R}^n$ and $0 < t \leq 2^{-n}$. Show that $A = \bigcup_n A_n$ is locally compact but its closure is not. Explain why this shows that the completion of a locally compact metric space is not necessarily locally compact. [*Hint:* The family $\{ A_n \}$ is a locally finite family of closed locally compact subspaces in A. Use this to show that the union is locally compact, and show that the closure of A contains all of $\ell^2 \times \{0\}$. Explain why ℓ^2 is not locally compact.]

SOLUTION.

As usual, we follow the hints.

> [*Show that*] *the family* $\{ A_n \}$ *is a locally finite family of closed locally compact subspaces in* A.

If $(x, t) \in \ell^2 \times (0, +\infty)$ and U is the open set $\ell^2 \times (t/2, +\infty)$, then $x \in U$ and $A_n \cap U = \emptyset$ unless $2^{-n} \geq t/2$, and therefore the family is locally finite. Furthermore, each set is closed in $\ell^2 \times (0, +\infty)$ and therefore also closed in $A = \bigcup_n A_n$.

> *Use this to show that the union is locally compact.*

We shall show that *if* A *is a* $\mathbf{T_3}$ *space that is a union of a locally finite family of closed locally compact subsets* A_α, *then* A *is locally compact.* Let $x \in A$, and let U be an open subset of A containing x such that $U \cap A_\alpha = \emptyset$ unless $\alpha = \alpha_1 \ \cdots \ , \alpha_k$. Let V_0 be an open subset of A such that $x \in V_0 \subset \overline{V_0} \subset U$, for each i choose an open set $W_i \in A_{\alpha_i}$ such that the closure of W_i is compact, express W_i as an intersection $V_i \cap A_{\alpha_i}$ where V_i is open in A, and finally let $V = \cap_i V_i$. Then we have

$$\overline{V} \ = \ \bigcup_{i=1}^{k} \left(\overline{V} \cap A_{\alpha_i} \right) \ \subset \ \bigcup_{i=1}^{k} \left(\overline{V_i} \cap A_{\alpha_i} \right) \ = \ \bigcup_{i=1}^{k} \mathbf{Closure} \left(W_i \ \text{ in } \ A_{\alpha_i} \right) .$$

Since the set on the right hand side is compact, the same is true for $\overline{V}$. Therefore we have shown that A is locally compact.

> *Show that the closure of* A *contains all of* $\ell^2 \times \{0\}$. *Explain why* ℓ^2 *is not locally compact.*

Let $x \in \ell^2$, and for each positive integer k let $P_k(x) \in A_k$ be the point $(H_k(x), 2^{-(k+1)})$, where $H_k(x)$ is the point whose first k coordinates are those of x and whose remaining coordinates are 0. It is an elementary exercise to verify that $(x, 0) = \lim_{k\to\infty} P_k(x)$. To conlclude we need to show that ℓ^2 is not locally compact. If it were, then there would be some $\varepsilon > 0$ such that the set of all $y \in \ell^2$ satisfying every $|y| \leq \varepsilon$ would be compact, and consequently infinite sequence $\{ y_n \}$ in ℓ^2 with $|y_n| \leq \varepsilon$ (for all n) would have a convergent subsequence. To see this does not happen, let $y_k = \frac{1}{2}\varepsilon\, \mathbf{e}_k$, where $\mathbf{e}_k$ is the k^{th} standard unit vector in ℓ^2. This sequence satifies the boundedness condition but does not have a convergent subsequence. Therefore ℓ^2 is not locally compact.■

FOOTNOTE.

Basic theorems from functional analysis imply that a normed vector space is locally compact if and only if it is finite-dimensional.■

5. Let X be a compact Hausdorff space, and let $U \subset X$ be open and noncompact. Prove that the collapsing map $c : X \to U^\bullet$ such that $c|U = \mathrm{id}_U$ and $c = \infty_U$ on $X - U$ is continuous. Show also that c is not necessarily open.

SOLUTION.

`To be inserted.`

6. **(a)** Explain why a compact Hausdorff space has no nontrivial Hausdorff abstract closures.

SOLUTION.

If X is compact Hausdorff and $f : X \to Y$ is a continuous map into a Hausdorff space, then $f(X)$ is closed. Therefore $f(X) = Y$ if the image of f is dense, and in fact f is a homeomorphism.■

(b) Prove that a Hausdorff space X has a maximal abstract Hausdorff closure that is unique up to equivalence. [*Hint:* Consider the identity map.]

SOLUTION.

This exercise shows that a formal analog of an important concept (the Stone-Čech compactification) is not necessarily as useful as the original concept itself; of course, there are also many situations in mathematics where the exact opposite happens. In any case, given an abstract closure (Y, f) we must have $(X, \mathrm{id}_X) \geq (Y, f)$ because $f : X \to Y$ trivially satisfies the condition $f = f \circ \mathrm{id}_X$.■

7. Suppose that X is compact Hausdorff and A is a closed subset of X. Prove that X/A is homeomorphic to the one point compactification of $X - A$.

SOLUTION.

See the footnote to Exercise 5 above.■

VI.5 : Metrization theorems

Problems from Munkres, § 40, p. 252

2. A subset W of X is said to be an $\mathbf{F}_\sigma$ set in X if W is a countable union of closet subsets of X. Show that W is an $\mathbf{F}_\sigma$ set in X if and only if $X - W$ is a $\mathbf{G}_\delta$ set in X. [The reason for the terminology is discussed immediately following this exercise on page 252 of Munkres.]

SOLUTION.

If W is an $\mathbf{F}_\sigma$ set then $W = \cup_n F_n$ where n ranges over the nonnegative integers and F_n is closed in X. Therefore

$$X - W \quad = \quad \bigcap_n X - F_n$$

is a countable intersection of the open subsets $X - F_n$ and accordingly is a $\mathbf{G}_\delta$ set.

Conversely, if $V = X - W$ is a $\mathbf{G}_\delta$ set, then $V = \cap_n U_n$ where n ranges over the nonnegative integers and U_n is open in X. Therefore

$$W \quad = \quad X - V \quad = \quad \bigcup_n X - U_n$$

is a countable union of the closed subsets $X - U_n$ and accordingly is an $\mathbf{F}_\sigma$ set.■

FOOTNOTE.

We have already shown that a closed subset of a metrizable space is a $\mathbf{G}_\delta$ set, and it follows that every open subset of a metrizable space is an $\mathbf{F}_\sigma$ set.■

3. Many spaces have *countable* bases, but no $\mathbf{T_1}$ space has a *locally finite* basis unless it is discrete. Prove this fact.

SOLUTION.

`To be inserted.`

Additional exercises

1. A *pseudometric space* is a pair $(X, \mathbf{d})$ consisting of a nonempty set X and a function $\mathbf{d}X \times X \to \mathbf{R}$ that has all the properties of a metric except possibly the property that $\mathbf{d}(u, v) = 0$.

(a) If ε-neighborhoods and open sets are defined as for metric spaces, explain why one still obtains a topology for pseudometric spaces.

SOLUTION.

None of the arguments verifying the axioms for open sets in metric spaces rely on the assumption $\mathbf{d}(x, y) = 0 \Longrightarrow x = y$.■

(b) Given a pseudometric space, define a binary relation $x \sim y$ if and only if $\mathbf{d}(x, y) = 0$. Show that this defines an equivalence relation and that $\mathbf{d}(x, y)$ only depends upon the equivalence classes of x and y.

SOLUTION.

The relation is clearly reflexive and symmetric. To see that it is transitive, note that $\mathbf{d}(x,y) = \mathbf{d}(y,z) = 0$ and the Triangle Inequality imply

$$\mathbf{d}(x,z) \;\; \leq \;\; \mathbf{d}(x,y) \; + \; \mathbf{d}(y,z) \;\; = \;\; 0 \; + \; 0 \;\; = \;\; 0 \; .$$

Likewise, if $x \sim x'$ and $y \sim y'$ then

$$\mathbf{d}(x',y') \;\; \leq \;\; \mathbf{d}(x',x) \; + \; \mathbf{d}(x,y) \; + \; \mathbf{d}(y,y') \;\; = \;\; 0 \; + \; \mathbf{d}(x,y) \; + \; 0 \;\; = \;\; \mathbf{d}(x,y)$$

and therefore the distance between two points only depends upon their equivalence classes with respect to the given relation.■

(c) Given a sequence of pseudometrics $\mathbf{d}_n$ on a set X, let $\mathbf{T}_\infty$ be the topology generated by the union of the sequence of topologies associated to these pseudometrics, and suppose that for each pair of distinct points $u\,v \in X$ there is some n such that $\mathbf{d}_n(u,v) > 0$. Prove that $(X, \mathbf{T}_\infty)$ is metrizable and that

$$\mathbf{d}_\infty \;\; = \;\; \sum_{n=1}^{\infty} \frac{\mathbf{d}_n}{2^n(1+\mathbf{d}_n)}$$

defines a metric whose underlying topology is $\mathbf{T}_\infty$.

SOLUTION.

First of all we verify that $\mathbf{d}_\infty$ defines a metric. In order to do this we must use some basic properties of the function

$$\varphi(x) \;\; = \;\; \frac{x}{1+x} \; .$$

This is a continuous and strictly increasing function defined on $[0, +\infty)$ and taking values in $[0,1)$ and it has the additional property $\varphi(x+y) \leq \varphi(x) + \varphi(y)$. The continuity and monotonicity properties of φ follow immediately from a computation of its derivative, the statement about its image follows because $x \geq 0$ implies $0 \leq \varphi(x) < 1$ and $\lim_{x \to +\infty} \varphi(x) = 1$ (both calculations are elementary exercises that are left to the reader).

The inequality $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ is established by direct computation of the difference $\varphi(x) + \varphi(y) - \varphi(x+y)$:

$$\frac{x}{1+x} \; + \; \frac{y}{1+y} \; - \; \frac{x+y}{1+x+y} \;\; = \;\; \frac{x^2y \; + \; 2\,xy \; + \; xy^2}{(1+x)\,(1+y)\,(1+x+y)}$$

This expression is nonnegative if x and y are nonnegative, and therefore one has the desired inequality for φ. Another elementary but useful inequality is $\varphi(x) \leq x$ if $x \geq 0$ (this is true because $1 \leq 1+x$). Finally, we note that the inverse to the continuous strictly monotonic function φ is given by

$$\varphi^{-1}(y) \;\; = \;\; \frac{y}{1-y} \; .$$

It follows that if $\mathbf{d}$ is a pseudometric then so is $\varphi \circ \mathbf{d}$ with the additional property that $\varphi \circ \mathbf{d} \leq 1$. More generally, if $\{\, a_n \,\}$ is a convergent sequence of nonnegative real numbers and $\{\, \mathbf{d}_n \,\}$ is a sequence of pseudometrics on a set X, then

$$\mathbf{d}_\infty \;\; = \;\; \sum_{n=1}^{\infty} \; a_n \cdot \varphi \circ \mathbf{d}_n < \sum_{n=1}^{\infty} \; a_n < \infty$$

also defines a pseudometric on X (write out the details of this!). In our situation $a_n = 2^{-n}$. Therefore the only thing left to prove about $\mathbf{d}_\infty$ is that it is positive when $x \neq y$. But in our situation if $x \neq y$ then there is some n such that $\mathbf{d}_n(x,y) > 0$, and the latter in turn implies that

$$2^{-n}\varphi\big(\,\mathbf{d}_n(x,y)\,\big) > 0$$

and since the latter is one summand in the infinite sum of nonnegative real numbers given by $\mathbf{d}_\infty(x,y)$ it follows that the latter is also positive. *Therefore $\mathbf{d}_\infty$ defines a metric on X.*

To prove that the topology $\mathcal{M}$ defined by this metric and the topology $\mathbf{T}_\infty$ determined by the sequence of pseudometrics are the same. Let N_α denote an α-neighborhood with respect to the $\mathbf{d}_\infty$ metric, and for each n let $N^{\langle n\rangle}_\beta$ denote a β-neighborhood with respect to the pseudometric $\mathbf{d}_n$. Suppose that N_ε is a basic open subset for $\mathcal{M}$ where $\varepsilon > 0$ and $x \in X$. Choose A so large that $n \geq A$ implies

$$\sum_{k=A}^{\infty} 2^{-k} \quad < \quad \frac{\varepsilon}{2}$$

Let W_x be the set of all z such that $\mathbf{d}_k(x,z) < \varepsilon/2$ for $1 \leq k < A$. Then W_x is the finite intersection of the $\mathbf{T}_\infty$-open subsets

$$W^{\langle k\rangle}(x) \quad = \quad \{\ z \in X \ \mid\ \mathbf{d}_k(x,z) < \varepsilon/2\ \}$$

and therefore W_k is also $\mathbf{T}_\infty$-open. Direct computation shows that if $y \in W_k$ then

$$\mathbf{d}_\infty(x,y) \quad = \quad \sum_{n=1}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \quad = \quad \sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \ + \ \sum_{n=A}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \quad <$$

$$\left(\sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,)\right) + \frac{\varepsilon}{2} \quad < \quad \left(\sum_{n=1}^{A-1} \frac{2^{-n}\,\varepsilon}{2}\right) + \frac{\varepsilon}{2} \quad < \quad \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad < \quad \varepsilon$$

so that $W_x \subset N_\varepsilon(x)$. Therefore, if U is open with respect to $\mathbf{d}_\infty$ we have

$$U \quad = \quad \bigcup_{x\in U} \{x\} \quad \subset \quad \bigcup_{x\in U} W_x \quad \subset \quad \bigcup_{x\in U} N_\varepsilon(x) \quad \subset \quad U$$

which shows that U is a union of $\mathbf{T}_\infty$-open subsets and therefore is $\mathbf{T}_\infty$-open. *Thus $\mathcal{M}$ is contained in $\mathbf{T}_\infty$.*

To show the reverse inclusion, consider an subbasic $\mathbf{T}_\infty$-open subset of the form $N^{\langle n\rangle}_\varepsilon(x)$. If y belongs to the latter then there is a $\delta > 0$ such that $N^{\langle n\rangle}_\delta(y) \subset N^{\langle n\rangle}_\varepsilon(x)$; without loss of generality we may as well assume that $\delta < 2^{-k}$. If we set

$$\eta(y) \quad = \quad 2^{-k}\,\varphi\big(\,\delta(y)\,\big)$$

then $\mathbf{d}_\infty(z,y) < \eta(y)$ and

$$2^{-k}\,\varphi\circ\mathbf{d}_k \quad \leq \quad \mathbf{d}_\infty$$

imply that $2^{-k}\,\varphi\circ\mathbf{d}_k(z,y) < \eta(y)$ so that $\varphi\circ\mathbf{d}_k(z,y) < 2^k\,\eta(y)$ and

$$\mathbf{d}_k(z,y) \quad < \quad \varphi^{-1}\big(\,2^k\,\delta(y)\,\big)$$

and by the definition of $\eta(y)$ the right hand side of this equation is equal to $\delta(y)$. Therefore if we set $W = N_\varepsilon^{\langle k \rangle}(x)$ then we have

$$W \quad = \quad N_\varepsilon^{\langle k \rangle}(x) \quad = \quad \bigcup_{y \in W} \{y\} \quad \subset \quad \bigcup_{y \in W} N_{\eta(y)}(y) \quad \subset \quad \bigcup_{y \in W} N_{\delta(y)}^{\langle k \rangle}(y) \quad \subset \quad N_\varepsilon^{\langle k \rangle}(x)$$

which shows that $N_\varepsilon^{\langle k \rangle}(x)$ belongs to $\mathcal{M}$. *Therefore the topologies* $\mathbf{T}_\infty$ *and* $\mathcal{M}$ *are equal.*■

(d) Let X be the set of all continuous real valued functions on the real line $\mathbf{R}$. Prove that X is metrizable such that the restriction maps from X to $\mathbf{BC}([-n, n])$ are uniformly continuous for all n. [*Hint:* Let $\mathbf{d}_n(f, g)$ be the maximum value of $|f(x) - g(x)|$ for $|x| \leq n$.]

SOLUTION.

Take the pseudometrics $\mathbf{d}_n$ as in the hint, and given $h \in X$ let $J^{\langle n \rangle}(h)$ be its restriction to $[-n, n]$. Furthermore, let $\|...\|_n$ be the uniform metric on $\mathbf{BC}\,([-n, n]\,)$, so that

$$\mathbf{d}_n(f, g) \quad = \quad \|\, J^{\langle n \rangle}(f) \;-\; J^{\langle n \rangle}(g)\,\|_n \;.$$

Given $\varepsilon > 0$ such that $\varepsilon < 1$, let $\delta = 2^{-n}\,\varphi(\varepsilon)$. Then $\mathbf{d}_\infty(f, g) < \delta$ implies $2^{-n}\,\varphi^\circ\mathbf{d}_n(f, g) < \delta$, which in turn implies $\mathbf{d}_n(f, g) < \varphi^{-1}(2^n\,\delta) = \varepsilon$.■

(e) Given X and the metric constructed in the previous part of the problem, prove that a sequence of functions $\{\, f_n \,\}$ converges to f if and only if for each compact subset $K \subset \mathbf{R}$ the sequence of restricted functions $\{\, f_n | K \,\}$ converges to $f | K$.

SOLUTION.

We first claim that $\lim_{n \to \infty} \mathbf{d}_\infty(f_n, f) = 0$ if and only if $\lim_{n \to \infty} \mathbf{d}_k(f_n, f) = 0$ for all k.

$(\implies)$ Let $\varepsilon > 0$ and fix k. Since $\lim_{n \to \infty} \mathbf{d}_\infty(f_n, f) = 0$ there is a positive integer M such that $n \geq M$ implies $\mathbf{d}_\infty(f_n, f) < 2^k\,\varphi(\varepsilon)$. Since

$$2^{-k}\varphi^\circ\mathbf{d}_k \quad \leq \quad \mathbf{d}_\infty$$

it follows that $\mathbf{d}_k \leq \varphi^{-1}(2^{-k}\,\mathbf{d}_\infty)$ and hence that $\mathbf{d}_k(f_n, f) < \varepsilon$ if $n \geq M$.■

$(\impliedby)$ Let $\varepsilon > 0$ and choose A such that

$$\sum_{k=n}^{\infty} 2^{-k} \quad < \quad \frac{\varepsilon}{2} \;.$$

Now choose B so that $n \geq B$ implies that

$$\mathbf{d}_k(f_n, f) \quad < \quad \frac{\varepsilon}{2A}$$

for all $k < A$. Then if $n \geq A + B$ we have

$$\mathbf{d}_\infty(f_n, f) \quad = \quad \sum_{k=1}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \quad = \quad \sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \sum_{k=A}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \quad <$$

$$\sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \frac{\varepsilon}{2} \quad < \quad \left(\sum_{k=1}^{A} 2^{-k}\,\frac{\varepsilon}{2A} \right) \;+\; \frac{\varepsilon}{2} \quad = \quad \varepsilon$$

so that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$.■

Now suppose that K is a compact subset of $\mathbf{R}$, and let $\|...\|_K$ be the uniform norm on $\mathbf{BC}(K)$. Then $K \subset [-n, n]$ for some n and thus for all $g \in X$ we have $\|g\|_K \leq \mathbf{d}_n(g, 0)$. Therefore if $\{ f_n \}$ converges to f then the sequence of restricted functions $\{ f_n|K \}$ converges uniformly to $f|K$. Conversely, if for each compact subset $K \subset \mathbf{R}$ the sequence of restricted functions $\{ f_n|K \}$ converges to $f|K$, then this is true in particular for $K = [-L, L]$ and accordingly $\lim_{n\to\infty} \mathbf{d}_L(f_n, f) = 0$ for all L. However, as noted above this implies that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ and hence that $\{ f_n \}$ converges to f.■

(f) Is X complete with respect to the metric described above? Prove this or give a counterexample.

SOLUTION.

The answer is **YES**, and here is a proof: Let $\{ f_n \}$ be a Cauchy sequence in X. Then if $K_m = [-m, m]$ the sequence of restricted functions $\{ f_n|K_m \}$ is a Cauchy sequence in $\mathbf{BC}(K_m)$ and therefore converges to a limit function $g_m \in \mathbf{BC}(K_m)$. Since $\lim_{n\to\infty} f_n|K_m(x) = g_m(x)$ for all $x \in K_m$ it follows that $p \leq m$ implies $g_m|K_p = g_p$ for all such m and p. Therefore if we define $g(x) = g_m(x)$ if $|x| < m$ then the definition does not depend upon the choice of m and the continuity of g_m for each m implies the continuity of g. Furthermore, by construction it follows that

$$\lim_{n\to\infty} \mathbf{d}_m(f_n, g) \quad = \quad \lim_{n\to\infty} \|\,(f_n|K_m) \; - \; g_m\,\| \quad = \quad 0$$

for all m and hence that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ by part **(e)** above.■

(g) Explain how the preceding can be generalized from continuous functions on $\mathbf{R}$ to continuous functions on an arbitrary open subset $U \subset \mathbf{R}^n$.

SOLUTION.

The key idea is to express U as an increasing union of bounded open subsets V_n such that $\overline{V_n} \subset V_{n+1}$ for all n. If U is a proper open subset of $\mathbf{R}^n$ let $F = \mathbf{R}^n - U$ (hence F is closed), and let V_m be the set of all points x such that $|x| < m$ and $\mathbf{d}_{[2]}(x, F_m) > 1/m$, where $\mathbf{d}_{[2]}$ denotes the usual Euclidean metric; if $U = \mathbf{R}^n$ let V_m be the set of all points x such that $|x| < m$. Since $y \in U$ if and only if $\mathbf{d}_{[2]}(y, F) = 0$, it follows that $U = \cup_m V_m$. Furthermore, since $y \in \overline{V_m}$ implies $|x| \leq m$ and $\mathbf{d}_{[2]}(x, F_n) \geq 1/n$ (why?), we have $\overline{V_m} \subset V_{m+1}$. Since $\overline{V_m}$ is bounded it is compact.

If f and g are continuous real valued functions on U, define $\mathbf{d}_n(f, g)$ to be the maximum value of $|f(x) - g(x)|$ on $\overline{V_m}$. In this setting the conclusions of parts **(d)** through **(f)** go through with only one significant modification; namely, one needs to check that every compact subset of U is contained in some V_m. To see this, note that K is a compact subset that is disjoint from the closed subset F, and therefore the continuous function $\mathbf{d}_{[2]}(y, F)$ assumes a positive minimum value c_1 on K and that there is a positive constant c_2 such that $y \in K$ implies $|y| \leq c_2$. If we choose m such that $m > 1/c_1$ and $m > c_2$, then K will be contained in V_m as required.■

FOOTNOTE.

Here is another situation where one encounters metrics defined by an infinite sequence of pseudometrics. Let Y be the set of all infinitely differentiable functions on $[0, 1]$, let D^k denote the operation of taking the k^{th} derivative, and let $\mathbf{d}_k(f, g)$ be the maximum value of $|D^k f \; - \; D^k g|$, where $0 \leq k < \infty$. One can also mix this sort of example with the one studied in the exercise; for instance, one can consider the topology on the set of infinitely differentiable functions on $\mathbf{R}$ defined by the countable family of pseudometrics $\mathbf{d}_{k,n}$ where $\mathbf{d}_{k,n}(f, g)$ is the maximum of $|D^k f \; - \; D^k g|$ on the closed interval $[-n, n]$.■

2. **(a)** Let U be an open subset of $\mathbf{R}^n$, and let $f : U \to \mathbf{R}^n$ be a continuous function such that $f^{-1}(\{0\})$ is contained in an open subset V such that $V \subset \overline{V} \subset U$. Prove that there is a continuous function g from $S^n \cong (\mathbf{R}^n)^\bullet$ to itself such that $g|V = f|V$ and $f^{-1}(\{0\}) = g^{-1}(\{0\})$. [*Hint:* Note that

$$(\mathbf{R}^n)^\bullet \; - \; \{0\} \;\; \cong \;\; \mathbf{R}^n$$

and consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \;\; \subset \;\; (\mathbf{R}^n)^\bullet \; - \; \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbf{R}^n)^\bullet \; - \; V$ with the same codomain? What happens if we try to piece this together with the original function f defined on U?]

SOLUTION.

We shall follow the steps indicated in the hint(s).

Note that

$$(\mathbf{R}^n)^\bullet \; - \; \{0\} \;\; \cong \;\; \mathbf{R}^n$$

This is true because the left hand side is homeomorphic to the complement of a point in S^n, and such a complement is homeomorphic to $\mathbf{R}^n$ via stereographic projection (which may be taken with respect to an arbitrary unit vector on the sphere).■

Consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \;\; \subset \;\; (\mathbf{R}^n)^\bullet \; - \; \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbf{R}^n)^\bullet \; - \; V$ with the same codomain?

In the first step we noted that the codomain was homeomorphic to $\mathbf{R}^n$, and the Tietze Extension Theorem implies that a continuous function from a closed subset A of a metric space X into $\mathbf{R}^n$ extends to all of X.■

What happens if we try to piece this together with the original function f defined on U?

If h is the function defined above, then we can piece h and $f|\overline{V}$ together and obtain a continuous function on all of $(\mathbf{R}^n)^\bullet$ if and only if the given functions agree on the intersection of the two closed subsets. This intersection is equal to $\overline{V} - V$, and by construction the restriction of h to this subset is equal to the restriction of f to this subset.■

(b) Suppose that we are given two continuous functions g and g' satisfying the conditions of the first part of this exercise. Prove that there is a continuous funtion

$$G : S^n \times [0,1] \longrightarrow S^n$$

such that $G(x,0) = g(x)$ for all $x \in S^n$ and $G(x,1) = g'(x)$ for all $x \in S^n$ (*i.e.*, the mappings g and g' are **homotopic**).

SOLUTION.

Let h and h' denote the associated maps from $(\mathbf{R}^n)^\bullet \; - \; V$ to $(\mathbf{R}^n)^\bullet \; - \; \{0\} \cong \mathbf{R}^n$ that are essentially given by the restrictions of g and g'. Each of these maps has the same restriction to

$\left(\overline{V} - V\right) \sqcup \{\infty\}$. Define a continuous mapping $H : ((\mathbf{R}^n)^\bullet \ - \ V) \times [0,1] \longrightarrow (\mathbf{R}^n)^\bullet \ - \ \{0\} \cong \mathbf{R}^n$ by $H(x,t) = t\, h'(x) + (1-t)\, h(x)$, and define F' on $\overline{V} \times [0,1]$ by $F(x,t) = f(x)$. As in the first part of this exercise, the mappings H and F' agree on the intersection of their domains and therefore they define a continuous map G on all of $S^n \times [0,1]$. Verification that $G(x,0) = g(x)$ and $G(x,1) = g'(x)$ is an elementary exercise.∎

3. Let X be compact, and let $\mathcal{F}$ be a family of continuous real valued functions on X that is closed under multiplication and such that for each $x \in X$ there is a neighborhood U of X and a function $f \in \mathcal{F}$ that vanishes identically on U. Prove that $\mathcal{F}$ contains the zero function.

SOLUTION.

For each x let φ_x be a continuous real valued function in $\mathcal{F}$ such that $\varphi_x = 0$ on some open set U_x containing x. The family of open sets $\{\, U_x \,\}$ is an open covering of X and therefore has a finite subcovering by sets $U_{x(i)}$ for $1 \leq i \leq k$. The product of the functions $\prod_i f_{x(i)}$ belongs to $\mathcal{F}$ and is zero on each $U_{x(i)}$; since the latter sets cover X it follows that the product is the zero function and therefore the latter belongs to $\mathcal{F}$.∎

4. Let X be a compact metric space, and let J be a nonempty subset of the ring $\mathbf{BC}(X)$ of (bounded) continuous functions on X such that J is closed under addition and subtraction, it is an *ideal* in the sense that $f \in J$ and $g \in \mathbf{BC}(X) \Longrightarrow f \cdot g \in J$, and for each $x \in X$ there is a function $f \in J$ such that $f(x) \neq 0$. Prove that $J = \mathbf{BC}(X)$. [*Hints:* This requires the existence of partitions of unity as established in Theorem 36.1 on pages 225–226 of Munkres; as noted there, the result works for arbitrary compact Hausdorff spaces, but we restrict to metric spaces because the course does not cover Urysohn's Lemma in that generality. Construct a finite open covering of X, say $\mathcal{U}$, such that for each $U_i \in \mathcal{U}$ there is a function $f_i \in J$ such that $f_i > 0$ on U_i. Let $\{\, \varphi_i \,\}$ be a partition of unity dominated by $\mathcal{U}$, and form $h = \sum_i \varphi_i \cdot f_i$. Note that $h \in J$ and $h > 0$ everywhere so that h has a reciprocal $k = 1/h$ in $\mathbf{BC}(X)$. Why does this imply that the constant function 1 lies in J, and why does the latter imply that everything lies in J?]

REMINDER.

If X is a topological space and $\mathcal{U} = \{U_1, \ \cdots \ , U_n\}$ is a finite indexed open covering of X, then a partition of unity subordinate to $\mathcal{U}$ is an indexed family of continuous functions $\varphi_i : X \to [0,1]$ for $1 \leq i \leq n$ such that for each i the zero set of the function φ_i contains contains $X - U_i$ in its interior and

$$\sum_{i=1}^{n} \varphi_i \quad = \quad 1 \ .$$

Theorem 36.1 on pages 225–226 of Munkres states that for each finite indexed open covering $\mathcal{U}$ of a $\mathbf{T_4}$ space (hence for each such covering of a compact Hausdorff space), there is a partition of unity subordinate to $\mathcal{U}$. The proof of this is based upon Urysohn's Lemma, so the methods in Munkres can be combined with our proof of the result for metric space to prove the existence of partitions of unity for indexed finite open coverings of compact metric spaces.

SOLUTION.

For each $x \in X$ we are assuming the existence of a continuous bounded function f_x such that $f_x(x) \neq 0$. Since J is closed under multiplication, we may replace this function by it square if necessary to obtain a function $g_x \in J$ such that $g_x(x) > 0$. Let U_x be the open set where g_x is nonzero, and choose an open subset V_x such that $\overline{V_x} \subset U_x$. The sets V_x determine an open covering of X; this open covering has a finite subcovering that we index and write as $\mathcal{V} = \{V_1, \ \cdots \ , V_n\}$. For each i let $g_i \in J$ be the previously chosen function that is positive on $\overline{V_i}$, and consider the function

$$g = \sum_{i=0}^{k} \varphi_i \cdot g_i \ .$$

This function belongs to J, and we claim that $g(y) > 0$ for all $y \in X$. Since $\sum_i \varphi_i = 1$ there is some index value m such that $\varphi_m(y) > 0$; by definition of a partition of unity this means that $y \in V_m$. But $y \in V_m$ implies $g_m(y) > 0$ too and therefore we have $g(y) \geq \varphi_m(y)\, g_m(y) > 0$. Since the reciprocal of a nowhere vanishing continuous real valued function on a compact space is continuous (and bounded!), we know that $1/g$ lies in $\mathbf{BC}(X)$. Since $g \in J$ it follows that $1 = g \cdot (1/g)$ also lies in J, and this in turn implies that $h = h \cdot 1$ lies in J for all $h \in \mathbf{BC}(X)$. Therefore $J = \mathbf{BC}(X)$ as claimed.■

5. In the notation of the preceding exercise, an ideal $\mathbf{M}$ in $\mathbf{BC}(X)$ is said to be a *maximal ideal* if it is a proper ideal and there are no ideals $\mathbf{A}$ such that $\mathbf{M}$ is properly contained in $\mathbf{A}$ and $\mathbf{A}$ is properly contained in in $\mathbf{BC}(X)$. Prove that there is a 1–1 correspondence between the maximal ideals of $\mathbf{BC}(X)$ and the points of X such that the ideal $\mathbf{M}_x$ corresponding to X is the set of all continuous functions $g : X \to \mathbf{R}$ such that $g(x) = 0$. [*Hint:* Use the preceding exercise.]

SOLUTION.

Let $\mathcal{M}$ be the set of all maximal ideals. For each point $x \in X$ we need to show that $\mathbf{M}_x \in \mathcal{M}$. First of all, verification that $\mathbf{M}_x$ is an ideal is a sequence of elementary computations (which the reader should verify). To see that the ideal is maximal, consider the function $\widehat{x} : \mathbf{BC}(X) \to \mathbf{R}$ by the formula $\widehat{x}(f) = f(x)$. This mapping is a ring homomorphism, it is onto, and $\widehat{x}(f) = 0$ if and only if $f \in \mathbf{M}_x$. Suppose that the ideal is not maximal, and let $\mathbf{A}$ be an ideal such that $\mathbf{M}_x$ is properly contained in $\mathbf{A}$. Let $a \in A$ be an element that is not in $\mathbf{M}_x$. Then $a(x) = \alpha \neq 0$ and it follows that $a(x) - \alpha 1$ lies in $\mathbf{M}_x$. It follows that $\alpha 1 \in \mathbf{A}$, and since $\mathbf{A}$ is an ideal we also have that $1 = \alpha^{-1}(\alpha 1)$ lies in $\mathbf{A}$; the latter in turn implies that every element $f = f \cdot 1$ of $\mathbf{BC}(X)$ lies in $\mathbf{A}$.

We claim that the map from X to $\mathcal{M}$ sending x to $\mathbf{M}_x$ is 1–1 and onto. Given distinct points x and y there is a bounded continuous function f such that $f(x) = 0$ and $f(y) = 1$, and therefore it follows that $\mathbf{M}_x \neq \mathbf{M}_y$. To see that the map is onto, let $\mathbf{M}$ be a maximal ideal, and note that the preceding exercise implies the existence of some point $p \in X$ such that $f(p) = 0$ for all $f \in \mathbf{M}$. This immediately implies that $\mathbf{M} \subset \mathbf{M}_p$, and since both are maximal (proper) ideals it follows that they must be equal. Therefore the map from X to $\mathcal{M}$ is a 1–1 correspondence.■

FOOTNOTES.

(1) One can use techniques from functional analysis and Tychonoff's Theorem to put a natural topology on $\mathcal{M}$ (depending only on the structure of $\mathbf{BC}(X)$ as a BAnach space and an algebra over the reals) such that the correspondence above is a homeomorphism; see page 283 of Rudin, *Functional Analysis*, for more information about this..

(2) The preceding results are the first steps in the proof of an important result due to I. Gelfand and M. Naimark that give a complete set of abstract conditions under which a Banach algebra is isomorphic to the algebra of continuous complex valued functions on a compact Hausdorff space. A Banach algebra is a combination of Banach space and associative algebra (over the real or complex numbers) such that the multiplication and norm satisfy the compatibility relation $|xy| \leq |x| \cdot |y|$. The additional conditions required to prove that a Banach algebra over the complex numbers is isomorphic to the complex version of $\mathbf{BC}(X)$ are commutativity, the existence of a unit element, and the existence of an conjugation-like map (formally, an **involution**) $a \to a^*$ satisfying the additional condition $|aa^*| = |a|^2$. Details appear in Rudin's book on functional analysis, and a reference for the Gelfand-Naimark Theorem is is Theorem 11.18 on page 289. A classic reference on Banach algebras (definitely not up-to-date in terms of current knowledge but an excellent source for the material it covers) is the book by Rickart in the bibliographic section of the course notes.