Examples of iterated subspace inclusions

The Schröder-Bernstein Theorem in set theory states that if there are injections from set A to set B and vice versa, then there is a bijection from A to B, and the Axiom of Choice yields a similar conclusion if "injection" is replaced by "surjection" (since in these cases one also has injections in the opposite directions). One of the exercises in Munkres involves finding counterexamples to a topological analog if one replaces injections by topological embeddings (injections that are homeomorphisms onto their images), and as noted in `math205Aupdate4.*` there are also counterexamples if one replaces surjections by closed quotient maps of compact connected metric spaces and bijections by homeomorphisms (in that example, the two spaces are $[0,1]$ and its Cartesian product with itself, which are not homeomorphic because the complements of one point subsets are always connected in the second case but not necessarily so in the first). In fact, one can construct counterexamples to the statement about injections using compact connected metric spaces, and the purpose of this document is to do so.

Let A be the solid unit disk in $\mathbf{R}^2$, and let B be the union of A with the line segment joining the origin to $(0,2)$. Then A and B are not homeomorphic for the same sorts of reasons as before (complements of one point subsets of A are always connected but not necessarily so in B). We have the obvious embedding from A to B by inclusion, and the map sending $b \in B$ to $\frac{1}{2}b$ defines a topological embedding in the opposite direction. Therefore A and B are spaces for which there exist topological embeddings in each direction, but A and B are not homeomorphic.■