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Mathematics 205A, Fall 2008, Final Examination

Answer Key

1. [*25 points*] Let X be a set with 2 or more elements. Show that there are topologies $\mathbf{U}$ and $\mathbf{V}$ on X such that the "identity map" $J : (X, \mathbf{U}) \to (X, \mathbf{V})$, defined by $J(x) = x$, is continuous, 1–1 and onto, but does not have a continuous inverse.

SOLUTION

Take $\mathbf{U}$ to be the discrete topology and $\mathbf{V}$ to be the indiscrete topology. Then the identity map J is continuous because every subset in $\mathbf{V}$ is also in $\mathbf{U}$, but the inverse is not because there are subsets in $\mathbf{U}$ which are not subsets in $\mathbf{V}$. More generally, the same considerations apply if $\mathbf{U}$ and $\mathbf{V}$ are two topologies on X such that $\mathbf{U}$ strictly contains $\mathbf{V}$.

2. [*25 points*] A subset A of a topological space X is said to be *locally closed* if $A = E \cap V$, where E is closed in X and V is open in X.

(a) Explain why the half-open interval $[0, 1)$ is a locally closed subset of the real numbers (with the usual topology) but is neither open nor closed.

(b) Let X be locally compact and Hausdorff, and suppose that A is a locally closed subset of X. Show that A is also locally compact and Hausdorff.

SOLUTION

(a) We can write $[0, 1) = [0, 1] \cap (-1, 1)$, so it is the intersection of an open subset and a closed subset. It is not closed in $\mathbb{R}$ because 1 is a limit point of $[0, 1)$ which does not lie in that set. It is not open, for if V is an open subset containing 0 then there is some $h > 0$ such that the open interval $(-h, h)$ is contained in V; in particular, it follows that V must have some points that are negative. Since all points in $[0, 1)$ are nonnegative and the set contains 0, it follows that the half-open interval cannot be an open subset of $\mathbb{R}$.

(b) Since every subspace of a Hausdorff space is Hausdorff, it follows immediately that A must be Hausdorff.

We know that a subset Y of a locally compact and Hausdorff space also has these properties if it is open or closed. If A is locally closed in X, then $A = E \cap U$, where E is closed in X and U is open in X. Since E is closed in X, it is locally compact and Hausdorff. Since $E \cap U$ is open in E and we know that the latter is locally compact and Hausdorff, it follows that $E \cap U$ must also be locally compact and Hausdorff.

3. [*25 points*] A $\mathbf{T_1}$ topological space X is dense in itself if every one point subset is nowhere dense in X; this is equivalent to saying that for all $y \in X$ the complement $X - \{y\}$ is dense in X (you may assume this).

(a) Suppose that the rational numbers $\mathbb{Q}$ have the subspace topology induced by the inclusion $\mathbb{Q} \subset \mathbb{R}$. Explain why $\mathbb{Q}$ is dense in itself.

(b) Explain why there is no complete metric on $\mathbb{Q}$ which defines the topology in (a).

SOLUTION

(a) Suppose that $q \in \mathbb{Q}$ and $\varepsilon > 0$. Then there is some $r \in \mathbb{Q}$ such that $q < r < q + \varepsilon$, and it follows if V is a nonempty open subset containing q, then $V - \{q\} \neq \emptyset$. On the other hand, if V is a nonempty open subset such that $q \notin V$, then clearly we still have $\emptyset \neq V = V - \{q\}$, so every nonempty open subset V has the property that $V - \{q\} = V \cap (\mathbb{Q} - \{q\}) \neq \emptyset$.

(b) The first half of the problem shows that for each rational number q the set $\{q\}$ is nowhere dense in $\mathbb{Q}$. Since the latter is countable, it follows that $\mathbb{Q}$ is a countable union of closed nowhere dense subsets. On the other hand, Baire's Theorem shows that such a union is never dense in a topological space which comes from a complete metric space. Therefore there is no complete metric on $\mathbb{Q}$ which defines the usual topology on $\mathbb{Q}$.

4. [*25 points*] Prove Lebesgue's Covering Lemma: If $(X, \mathbf{d})$ is a compact metric space and $\mathcal{U}$ is an open covering of X, then there is a positive real number ε such that if $u, v \in X$ satisfy $\mathbf{d}(u,v) < \varepsilon$, then there is some $W \in \mathcal{U}$ such that $u, v \in W$.

SOLUTION

Let $\mathcal{U}$ be an open covering of X. For each $x \in X$ there is a some $\varepsilon(x) > 0$ such that $V_x = N_{\varepsilon(x)}(x)$ is contained in some element of $\mathcal{U}$. Let $W_x = N_{\varepsilon(x)/2}(x)$. The collection $\mathcal{W}$ of all such sets W_x forms an open covering of X, so it has a finite subcovering $\{W_1, \cdots, W_n\}$ associated to points $x_1, \cdots, x_n \in X$. Let

$$\varepsilon \quad = \quad \min_k \frac{\varepsilon(x_k)}{2} \ .$$

Suppose that $u, v \in X$ and $\mathbf{d}(u,v) < \varepsilon$. Then $u \in W_k$ for some k, and $v \in V_k$ because we have

$$\mathbf{d}(v, x_k) \ \leq \ \mathbf{d}(u, x_k) + \mathbf{d}(u,v) \ < \ \frac{\varepsilon_k}{2} + \varepsilon \ \leq$$

$$\frac{\varepsilon_k}{2} + \frac{\varepsilon_k}{2} \ = \ \varepsilon_k \ .$$

Thus we have $u, v \in V_k$, and since V_k is contained in some open set from $\mathcal{U}$, it follows that u and v are contained in some open set from $\mathcal{U}$.

5. [*25 points*] If X and Y are first countable spaces, prove that $X \times Y$ with the product topology is first countable.

SOLUTION

Let $(x, y) \in X \times Y$, let $\mathcal{U}_x = \{ U_1 \supset U_2 \ \cdots \}$ and $\mathcal{V}_y = \{ V_1 \supset V_2 \ \cdots \}$ be countable neighborhood bases at x and y respectively, and suppose that W is an open neighborhood of (x, y) in $X \times Y$. By the construction of the product topology, there are open subsets $U' \subset X$ and $V' \subset Y$ containing x and y respectively such that $(x, y) \in U' \times V' \subset W$. By first countability there open subsets $U_p \in \mathcal{U}$ and $V_q \in \mathcal{V}$ such that $x \in U_p \subset U'$ and $y \in V_q \subset V'$. If we take n to be the larger of p and q, it follows that $(x, y) \in U_n \times V_n \subset U' \times V' \subset W$, and hence the family

$$U_1 \times V_1 \quad \supset \quad U_2 \times V_2 \quad \supset \quad \cdots$$

is a countable neighborhood base at (x, y), so that the product is first countable.

6. [*25 points*] In each of the following, give an example of a topological space X with the stated properties. You need not give reasons for your answers, but explanations might yield partial credit if the examples are not correct.

(a) A space X which has a connected component that is not an open subset.

(b) A space X which is locally connected but not connected.

(c) A space X which is locally connected and connected.

(d) A space X which is connected but not arcwise connected.

SOLUTION

(a) Take the set of all real numbers expressible as $\frac{1}{n}$ where n is a positive integer together with 0. Then all the components are one points sets, but $\{0\}$ is not open in this set.

(b) Take a disjoint union of two open intervals in $\mathbb{R}$ like $(0,1) \cup (2,3)$.

(c) The real numbers satisfy both conditions.

(d) Take the closure (in $\mathbb{R}^2$) of the graph of $y = \sin(1/x)$, where $x > 0$, whose arc components are the graph itself and the closed segment $\{0\} \times [-1,1]$.

7. [*25 points*] Suppose that X is a topological space and $A \subset X$ is nonempty and closed. Define an equivalence relation on X whose equivalence classes are A together with one point sets $\{y\}$ where $y \in X - A$, let X/A denote the set of equivalence classes with the quotient topology, and let $p : X \to X/A$ be the usual (continuous) quotient projection. Prove that p is a closed mapping. [*Hint:* Given a closed subset $F \subset X$, there are two cases depending upon whether or not $F \cap A$ is empty or nonempty.]

SOLUTION

We need to show that for each closed subset F, its image is closed in X/A. Equivalently, by the definition of the quotient topology we need to show that if F is closed in X, then the set

$$E \quad = \quad p^{-1}\big[\, p[F] \,\big]$$

is closed in X. Following the hint, we have two cases depending upon whether or not $F \cap A$ is empty. If $F \cap A = \emptyset$, then E is just F itself and hence it is closed. On the other hand, it $F \cap A \neq \emptyset$, then it follows that $E = F \cup A$; since this is a union of closed subsets, it is also closed. Now this exhausts all possible cases, so it follows that p is a closed mapping.

8. [*25 points*] Suppose that $A_1, \cdots, A_n$ is a finite sequence of arcwise connected subset of a topological space X such that $A_{i+1} \cap A_i \neq \emptyset$ for $1 \leq i < n$. Prove that the union $\cup_i A_i$ of the A_i is arcwise connected. [*Hint:* Induction on n.]

SOLUTION

Note first that the assumptions imply each set A_i is nonempty. If $n = 1$ the result is trivial because we already know that A_1 is arcwise connected. Suppose now that the result is true for $n \geq 1$ and we are given a family of $n + 1$ sets A_i with the stated properties. Let B_n be the union of A_1 through A_n; then the induction hypothesis implies that B_n is arcwise connected. Since $A_{n+1} \cap B_n$ contains $A_{n+1} \cap A_n$ and the latter is nonempty, it follows that the union $A_{n+1} \cup B_n$ is also arcwise connected. But the union is equal to $\cup_{i \leq n+1} A_i$, and hence the latter is arcwise connected, completing the proof of the inductive step in the argument.

9. [*30 points*] Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous mapping such that f is nowhere locally constant; in other words, for every point $x \in X$ and every open neighborhood U of x the restriction $f|U$ is not a constant mapping. Prove that if $x \in X$, then $f(x)$ is a limit point of $f[X]$.

SOLUTION

Let V be an open neighborhood of $f(x)$ in Y. By continuity $U = f^{-1}[V]$ is an open set containing y. Since f is nowhere locally constant, there is some $z \in U$ such that $f(z) \neq f(y)$. By the definition of U, it follows that $f(z) \in V$. Therefore $f(z) \in (V - \{f(x)\}) \cap f[X]$, so that the latter is nonempty, and this means that $f(x)$ must be a limit point of $f[X]$.