NAME: ________________________________

Mathematics 205A, Fall 2008, Examination 1

Answer Key

1. [*30 points*] Let $(X, \mathbf{d})$ be a metric space, let $x \in X$, let $r > 0$, and let $N_r(x)$ denote the set of all points $y \in X$ such that $\mathbf{d}(x, y) < r$.

(a) Prove that the map $f(y) = \mathbf{d}(x, y)$ is uniformly continuous.

(b) Prove that the set $F_r(x)$ of all $y \in X$ such that $\mathbf{d}(x, y) \leq r$ is a closed subset of X.

(c) Give an example such that $F_r(x)$ is not the closure of $N_r(x)$. [*Hint:* There are examples with finitely many points.]

SOLUTION

(a) We have $|f(u) - f(v)| \leq \mathbf{d}(u, v)$ by a standard consequence of the Triangle Inequality. Therefore if $\varepsilon > 0$ then $\mathbf{d}(u, v) < \varepsilon$ implies $|f(u) - f(v)| < \varepsilon$.$\blacksquare$

(b) If f is as in the first part of the problem, then $F_r(x) = f^{-1}\big[\,[0, r]\,\big]$, and by the continuity of f and the fact that $[0, r]$ is closed we know that $F_r(x)$ is closed.$\blacksquare$

(c) Take $X = \{0, 1\} \subset \mathbb{R}$. Then $N_1(0)$ is the closed subset $\{0\}$, but $F_1(0) = X$.$\blacksquare$

2. [*25 points*] (a) Let U be an open subset of a metric space X, and let $F \subset X$ be finite. Explain why $U - F$ is open in X.

(b) Suppose that $A \subset X$ as above, and that x is a limit point of A. Prove that for every $\varepsilon > 0$ there are infinitely many points in $A \cap N_\varepsilon(x)$.

SOLUTION

(a) Every one point subset of a metric space is closed, and therefore every finite set is also closed because each such set is a finite union of one point subsets. Therefore $X - F$ is open in X, so that $U - F = U \cap (X - F)$ is also open./bull

(b) Suppose that $A \cap N_\varepsilon(x)$ is a finite set F. Let $\delta > 0$ be the minimum of all distances $\mathbf{d}(y, x)$, where $y \in F$ and $y \neq x$. It then follows that $A \cap \big(N_{\delta/2}(x) - \{x\} \big)$ must be empty. On the other hand, but the definition of a limit point this intersection is nonempty, so we have a contradiction. The source of this contradiction is our assumption that $A \cap N_\varepsilon(x)$ is finite, and hence it follows that the latter must be infinite.$\blacksquare$

3. [*20 points*] Given two spaces A_1 and A_2, let $p_i : A_1 \times A_2$ denote the projection onto A_i for $i = 1, 2$. — Suppose now that we are given continuous mappings $f_i : X_i \to Y_i$ for $i = 1, 2$. Prove that the mapping $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ defined by $f_1 \times f_2(x_1, x_2) = \big(f_1(x_1),\ f_2(x_2)\big)$ is continuous. [*Hint:* Check that its projection onto Y_i is $f_i \circ p_i$.]

SOLUTION

A map F into a product of two spaces $Y_1 \times Y_2$ is continuous if and only if the coordinate projections $p^{Y_1} \circ f$ and $p^{Y_2} \circ f$ are continuous. But the defining identity for $F = f_1 \times f_2$ implies that

$$p^{Y_i} \circ f \quad = \quad p^{Y_i}\big(f_1(x_1),\ f_2(x_2)\big) \quad = \quad f_i(x_i) \quad = \quad f_i \circ p^{X_i}(x_1, x_2)$$

and the map on the right side is continuous because the maps f_i and the coordinate projections are all continuous. This shows that the coordinate projections of $f_1 \times f_2$ are continuous, and as noted above this means that $f_1 \times f_2$ itself is also continuous.■

4. [*10 points*] Give an example of a complete metric space which is not compact.

SOLUTION

The real number system is one example, and an infinite set with the discrete metric is another. The first one is not compact because a compact metric space is bounded, and the second is not because the infinite covering by one point (open) subsets has no finite subcovering (every finite union of sets in the family is finite).■

5. [*20 points*] Let X and Y be compact metric spaces, and suppose that $f : X \to Y$ is continuous, 1–1 and onto. Prove that f is a homeomorphism.

SOLUTION

It suffices to show that f sends closed subsets of X to closed subsets of Y. But if E is closed in the compact space X, then E is compact, so that its image $f[E]$ is also a compact subset of Y. However, compact subsets of a metric space are closed, and therefore $f[E]$ is closed in Y, which is what we needed to prove in order to show that f is a homeomorphism.■