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Mathematics 205A, Fall 2008, Examination 2

Answer Key

1. [*25 points*] Let A and B be subsets of complete metric spaces X and Y respectively, and suppose that $h : A \to B$ is a surjective isometry.

(i) Explain why the inclusion maps $j_A : A \to \overline{A}$ and $j_B : B \to \overline{B}$ are abstract completions.

(ii) Prove that there is a unique extension of h to a surjective isometry $H : \overline{A} \to \overline{B}$.

SOLUTION

For the first part, the axioms for a completion are that one has an isometry from the metric space M into the complete metric space N such that the image of M is dense in N. Since a closed subset of a complete metric space is complete, this holds if M is a subset of a complete metric space Z and N is the closure of M in Z, where the map from M to N is inclusion.

For the second part, the uniqueness result for completions implies that an isometry from one metric space onto another has a unique extension to an isometry on their respective completions.

2. [*15 points*] The theory of spherical coordinates implies that every point on the unit sphere $S^2 \subset \mathbb{R}^3$ (defined by $|\mathbf{x}| = 1$) can be written in the form

$$(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

for suitable real numbers θ and ϕ, and conversely every point of this form lies on S^2. Let $f : S^2 \to [0, 1]$ be a continuous function whose maximum and minimum values are 1 and 0 respectively. Explain why there is a point $\mathbf{x}$ on S^2 such that $f(\mathbf{x}) = \frac{1}{3}$.

SOLUTION

The spherical coordinate map from $\mathbb{R}^2$ to S^2 is continuous and onto, and therefore S^2 is connected. Therefore the image of S^2 under f is a connected subset of $\mathbb{R}$, which contains its upper bound of 1 and its lower bound of 0. Since connected subsets of $\mathbb{R}$ have the intemediate value property, this means that the image of f is the entire interval $[0, 1]$, and hence in particular it contains the point $\frac{1}{3}$.

3. [*20 points*] Suppose that X is a topological space and $A \subset X$ is connected and dense. Prove that X is connected.

SOLUTION

If A is empty, then X must also be empty and there is nothing to prove, so assume A is nonempty. Let E be a nonempty open closed subset of X. Then $X - E$ is also open and closed, and one of them contains a point $a_0 \in A$; without loss of generality we might as well assume $a_0 \in E$.

Now $A \cup E$ is a nonempty open closed subset of A, so by connectedness we have $A \cap E = A$, so that $A \subset E$. Since E is closed, we must also have $X = \overline{A} \subset E$. Therefore X must be connected.

4. [*15 points*] Let $U \subset \mathbb{R}^n$ be an open connected subset. Prove that U is arcwise connected.

SOLUTION

Let $\mathcal{A}$ be the equivalence relation whose equivalence classes are the arc components of U, and let C be an equivalence class of $\mathcal{A}$. Since U is open in $\mathbb{R}^n$, for each $x \in C$ there is some $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$, and since the ε-neighborhood is convex we know that such that $N_\varepsilon(x) \subset C$. Therefore C is open; since C is arbitrary, all equivalence classes are open.

Since the complement of one equivalence class is the union of the other equivalence classes, it also follows that C is closed, and hence each arc component is both open and closed. By connectedness this means $C = X$, so that there is only one arc component and hence X is arcwise connected.

5. [*25 points*] Let X be a separable metric space, and suppose that $A \subset X$. Prove that A is also separable.

SOLUTION

If X is a separable metric space, then X is second countable. Therefore the subspace A is also second countable. Since second countable topological spaces are separable, it follows that A must be separable. [*Reminder:* There are may examples of separable spaces which do not come from metric spaces and have subsets that are not separable.]