# SOLUTIONS TO EXERCISES FOR

# MATHEMATICS 205A — Part 6

**Fall 2008**

# APPENDICES

### Appendix A : Topological groups

(Munkres, Supplementary exercises following $ 22; *see also* course notes, Appendix D)

*Problems from Munkres,* § 30*, pp.* 194 – 195

Munkres, § 26, pp. 170–172: 12, 13

Munkres, § 30, pp. 194–195: 18

Munkres, § 31, pp. 199–200: 8

Munkres, § 33, pp. 212–214: 10

*Solutions for these problems will not be given.*

*Additional exercises*

**Notation.** Let $\mathbb{F}$ be the real or complex numbers. Within the matrix group $\mathbf{GL}(n, \mathbb{F})$ there are certain subgroups of particular importance. One such subgroup is the *special linear group* $\mathbf{SL}(n, \mathbb{F})$ of all matrices of determinant 1.

**0.** Prove that the group $\mathbf{SL}(2, \mathbb{C})$ has no nontrivial proper normal subgroups except for the subgroup $\{\pm I\}$. [*Hint:* If $N$ is a normal subgroup, show first that if $A \in N$ then $N$ contains all matrices that are similar to $A$. Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \qquad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here $\alpha$ is a complex number not equal to 0 or $\pm 1$ and $\varepsilon = \pm 1$. The idea is to show that if $N$ contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.]

SOLUTION.

We shall follow the steps indicated in the hint.

*If $N$ is a normal subgroup of $\mathbf{SL}(2, \mathbb{C})$ and $A \in N$ then $N$ contains all matrices that are similar to $A$.*

If $B$ is similar to $A$ then $B = P\,A\,P^{-1}$ for some $P \in \mathbf{GL}(2,\mathbb{C})$, so the point is to show that one can choose $P$ so that $\det P = 1$. The easiest way to do this is to use a matrix of the form $\beta P$ for some nonzero complex number $\beta$, and if we choose the latter so that $\beta^2 = (\det P)^{-1}$ then $\beta P$ will be a matrix in in $\mathbf{SL}(2,\mathbb{C})$ and $B$ will be equal to $(\beta P)\,A\,(\beta P)^{-1}$.■

*Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:*

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

*Here $\alpha$ is a complex number not equal to 0 or $\pm 1$ and $\varepsilon = \pm 1$.*

The preceding step that if $N$ contains a given matrix then it contains all matrices similar to that matrix. Therefore if $N$ is a nontrivial normal subgroup of $\mathbf{SL}(2,\mathbb{C})$ then $N$ is a union of similarity classes, and the latter in turn implies that $N$ contains a given matrix $A$ if and only if it contains a Jordan form for $A$. For $2 \times 2$ matrices there are only two basic Jordan forms; namely, diagonal matrices and elementary $2 \times 2$ Jordan matrices of the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda \neq 0$. Such matrices have determinant 1 if and only if the product of the diagonal entries is the identity, and this means that the Jordan forms must satisfy conditions very close in the hint; the only difference involves the diagonal case where the diagonal entries may be equal to $\pm 1$. If $N$ is neither trivial nor equal to the subgroup $\{\pm I\}$, then it must contain a Jordan form given by a diagonal matrix whose nonzero entries are not $\pm 1$ or else it must contain an elementary $2 \times 2$ Jordan matrix.

*The idea is to show that if $N$ contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.*

The basic strategy is to show first that if $N$ contains either of the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(where $\alpha \neq 0,\ \pm 1$), then $N$ also contains

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and next to show that if $N$ contains the latter matrix then $N$ contains every Jordan form. As noted before, the latter will imply that $N = \mathbf{SL}(2,\mathbb{C})$.

At this point one needs to do some explicit computations to find products of matrices in $N$ with sufficiently many Jordan forms. Suppose first that $N$ contains the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

Then $N$ contains $A^2$, which is equal to

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and the latter is similar to

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so $N$ contains $B$. Suppose now that $N$ contains some diagonal matrix A whose nonzero entries are $\alpha$ and $\alpha^{-1}$ where $\alpha \neq \pm 1$, and let $B$ be given as above. Then $N$ also contains the commutator $C = A\,B\,A^{-1}\,B^{-1}$. Direct computation shows that the latter matrix is given as follows:

$$C = \begin{pmatrix} 1 & \alpha^2 - 1 \\ 0 & 1 \end{pmatrix}$$

Since $\alpha \neq \pm 1$ it follows that $C$ is similar to $B$ and therefore $B \in N$.

We have now shown that if $N$ is a nontrivial normal subgroup that is not equal to $\{\pm I\}$, then $N$ must contain $B$. As noted before, the final step is to prove that if $B \in N$ then $N = \mathbf{SL}(2, \mathbb{C})$.

Since $B \in N$ implies that $N$ contains all matrices similar to $B$ it follows that all matrices of the forms

$$P = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad , \qquad Q = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

(where $z$ is an arbitrary complex number)

also belong to $N$, which in turn shows that $P\,Q \in N$. But.

$$P\,Q = \begin{pmatrix} 1 & z \\ z & z^2 + 1 \end{pmatrix}$$

and its characteristic polynomial is $t^2 - (z^2 + 2)\,t - 1$. If $z \neq 0$ then this polynomial has distinct nonzero roots such that one is the reciprocal of the other. Furthermore, if $\alpha \neq 0$ then one can always solve the equation $\alpha + \alpha^{-1} = z^2 + 2$ for $z$ over the complex numbers and therefore we see that for each $\alpha \neq 0$ there is a matrix $P\,Q$ similar to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

such that $P\,Q \in N$. It follows that each of the displayed diagonal matrices also lies in $N$; in particular, this includes the case where $\alpha = -1$ so that the diagonal matrix is equal to $-I$. To complete the argument we need to show that $N$ also contains a matrix similar to

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

But this is relatively easy because $N$ must contain $-B = (-I)B$ and the latter is similar to $A$.■

FOOTNOTE.

More generally, if $\mathbb{F}$ is an arbitrary field then every proper normal subgroup of $\mathbf{SL}(n, \mathbb{F})$ is contained in the central subgroup of diagonal matrices of determinant 1; of course, the latter is isomorphic to the group of $n^{\text{th}}$ roots of unity in $\mathbb{F}$.

**Definition.** The *orthogonal group* $\mathbf{O}(n)$ consists of all transformations in $\mathbf{GL}(n, \mathbb{R})$ that take each orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis. It is an easy exercise in linear algebra to show that the determinant of all matrices in $\mathbf{O}(n)$ is $\pm 1$. The *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ consisting of all matrices whose determinants are equal to $+1$. Replacing If we replace the real numbers $\mathbb{R}$ by the complex numbers $\mathbb{C}$ we get the unitary groups $\mathbf{U}(n)$ and the *special unitary groups* $\mathbf{SU}(n)$, which are the subgroups of $\mathbf{U}(n)$ given by matrices with determinant 1. The determinant of every matrix in $\mathbf{U}(n)$ is of absolute value 1 just as before, but in the complex case this means that the determinant is a complex number on the unit circle. In Appendix A the orthogonal and unitary groups were shown to be compact.

**1.** Show that $\mathbf{O}(1)$ is isomorphic to the cyclic group of order 2 and that $\mathbf{SO}(2)$ is isomorphic as a topological group to the circle group $S^1$. Conclude that $\mathbf{O}(2)$ is homeomorphic as a space to $\mathbf{SO}(2) \times \mathbf{O}(1)$, but that as a group these objects are not isomorphic to each other. [*Hint:* In Appendix D it is noted that every element of $\mathbf{O}(2)$ that is not in $\mathbf{SO}(2)$ has an orthonormal basis of eigenvectors corresponding to the eigenvalues $\pm 1$. What does this say about the orders of such group elements?]

SOLUTION.

The group $\mathbf{GL}(n, \mathbb{R})$ is isomorphic to the multiplicative group of nonzero real numbers. Therefore $\mathbf{O}(1)$ is isomorphic to set of nonzero real numbers that take 1 to an element of absolute value 1; but a nonzero real number has this property if and only if its absolute value is 1, or equivalently if it is equal to $\pm 1$.

Consider now the group $\mathbf{SO}(2)$. Since it sends the standard basis into an orthonormal basis, it follows that its columns must be orthonormal. Therefore there is some $\theta$ such that the entries of the first column are $\cos\theta$ and $\sin\theta$, and there is some $\varphi$ such that the entries of the second column are $\cos\varphi$ and $\sin\varphi$. Since the vectors in question are perpendicular, it follows that $|\theta - \varphi| = \pi/2$, and depending upon whether the sign of $\theta - \varphi$ is positive or negative there are two possibilities:

(1) If $\theta - \phi > 0$ then $\cos\varphi = -\sin\theta$ and $\sin\varphi = \cos\theta$.

(2) If $\theta - \phi < 0$ then $\cos\varphi = \sin\theta$ and $\sin\varphi = -\cos\theta$.

In the first case the determinant is 1 and in the second it is $-1$.

We may now construct the isomorphism from $S^1$ to $\mathbf{SO}(2)$ as follows: If $z = x + y\,i$ is a complex number such that $|z| = 1$, send $z$ to the matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} .$$

This map is clearly 1–1 because one can retrieve $z$ directly from the entries of the matrix. It is onto because the complex number $\cos\theta + i\,\sin\theta$ maps to the matrix associated to $\theta$. Verification that the map takes complex products to matrix products is an exercise in bookkeeping.

A homeomorphism from $\mathbf{SO}(2) \times \mathbf{O}(1)$ to $\mathbf{O}(2)$ may be constructed as follows: Let $\alpha : \mathbf{O}(1) \to \mathbf{O}(2)$ be the map that sends $\pm 1$ to the diagonal matrix whose entries are $\pm 1$ and 1, and define $M(A, \varepsilon)$ to be the matrix product $A\,\alpha(\varepsilon)$. Since the image of $-1$ does not lie in $\mathbf{SO}(2)$, standard results on cosets in group theory imply that the map $M$ is 1–1 and onto. But it is also continuous, and since it maps a compact space to a Hausdorff space it is a homeomorphism onto its image, which is $\mathbf{SO}(2)$.

To see that this map is not a group isomorphism, note that in the direct product group $\mathbf{SO}(2) \times \mathbf{O}(1)$ there are only finitely many elements of order 2 (specifically, the first coordinate must be $\pm I$). On the other hand, the results from Appendix D show that all elements of $\mathbf{O}(2)$ that are not in $\mathbf{SO}(2)$ have order 2 and hence there are infinitely many such element in $\mathbf{O}(2)$. Therefore the latter cannot be isomorphic to the direct product group.■

**2.** Show that $\mathbf{U}(1)$ is isomorphic to the circle group $S^1$ as a topological group.

SOLUTION.

This is similar to the first part of the preceding exercise. The group $\mathbf{GL}(n, \mathbb{C})$ is isomorphic to the multiplicative group of nonzero complex numbers. Therefore $\mathbf{U}(1)$ is isomorphic to set of nonzero complex numbers that take 1 to an element of absolute value 1; but a nonzero complex number has this property if and only if its absolute value is 1, or equivalently if it lies on the circle $S^1$.■

**3.** For each positive integer $n$, show that $\mathbf{SO}(n)$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are connected spaces and that $\mathbf{U}(n)$ is homeomorphic as a space (but not necessarily isomorphic as a topological group) to $\mathbf{SU}(n) \times S^1$. [*Hints:* In the complex case use the Spectral Theorem for normal matrices to show that every unitary matrix lies in the path component of the identity, which is a normal subgroup. In the real case use the results on normal form in Appendix D.]

SOLUTION.

The proof separates into three cases depending upon whether the group in question is $\mathbf{U}(n)$, $\mathbf{SU}(n)$ or $\mathbf{SO}(n)$,

*The case* $G = \mathbf{U}(n)$.

We start with the unitary group. The Spectral Theorem states that if $A$ is an $n \times n$ unitary matrix then there is another invertible unitary matrix $P$ such that $B = P\,A\,P^{-1}$ is diagonal. We claim that there is a continuous curve joining $B$ to the identity. To see this, write the diagonal entries of $B$ as $\exp it_j$ where $t_j$ is real and $1 \leq j \leq n$. Let $C(s)$ be the continuous curve in the diagonal unitary matrices such that the diagonal entries of $C(s)$ are $\exp(i\,st_j)$ where $s \in [0,1]$. It follows immediately that $C(0) = I$ and $C(1) = B$. Finally, if we let $\gamma(s) = P^{-1}\,C(s)\,P$ then $\gamma$ is a continuous curve in $\mathbf{U}(n)$ such that $\gamma(0) = I$ and $\gamma(1) = A$. This shows that $\mathbf{U}(n)$ is in fact arcwise connected.■

*The case* $G = \mathbf{U}(n)$.

The preceding argument also shows that $\mathbf{SU}(n)$ is arcwise connected; it is only necessary to check that if $A$ has determinant 1 then everything else in the construction also has this property. The determinants of $B$ and $A$ are equal because the determinants of similar matrices are equal. Furthermore, since the determinant of $B$ is 1 it follows that $\sum_j\ t_j = 0$, and the latter in turn implies that the image of the curve $C(s)$ is contained in $\mathbf{SU}(n)$. Since the latter is a normal subgroup of $\mathbf{U}(n)$ it follows that the curve $\gamma(s)$ also lies in $\mathbf{SU}(n)$, and therefore we conclude that the latter group is also arcwise connected.

The product decomposition for $\mathbf{U}(n)$ is derived by an argument similar to the previous argument for $\mathbf{O}(2)$. More generally, suppose we have a group $G$ with a normal subgroup $K$ and a second subgroup $H$ such that $G = H \cdot K$ and $H \cap K = \{1\}$. Then group theoretic considerations yield a 1–1 onto map $\varphi : K \times H \to G$ given by $\varphi(k,h) = k \cdot h$. If $G$ is a topological group and the subgroups have the subspace topologies then the 1–1 onto map $\varphi$ is continuous. Furthermore, if $G$ is compact Hausdorff and $H$ and $K$ are closed subgroups of $G$ then $\varphi$ is a homeomorphism. In our particular situation we can take $G = \mathbf{U}(n)$, $K = \mathbf{SU}(n)$ and $H \cong \mathbf{U}(1)$ to be the subgroup of diagonal matrices with ones down the diagonal except perhaps for the $(1,1)$ entry. These subgroups satisfy all the conditions we have imposed and therefore we have that $G$ is homeomorphic to the product $K \times H$.■

*The case $G = \mathbf{SO}(n)$.*

Finally, we need to verify that $\mathbf{SO}(n)$ is arcwise connected, and as indicated in the hint we use the normal form obtained in Appendix D. According to this result, for every orthogonal $n \times n$ matrix $A$ there is another orthogonal matrix $P$ such that $B = PAP^{-1}$ is is a block sum of orthogonal matrices that are either $2 \times 2$ or $1 \times 1$. These matrices may be sorted further using their determinants (recall that the determinant of an orthogonal matrix is $\pm 1$).

In fact, one can choose the matrix $P$ such that the block summands are sorted by size and determinant such that

(1) the $1 \times 1$ summands with determinant 1 come first,
(2) the $2 \times 2$ summands with determinant 1 come second,
(3) the $2 \times 2$ summands with determinant $-1$ come third,
(4) the $1 \times 1$ summands with determinant $-1$ come last.

Each matrix of the first type is a rotation matrix $R_\theta$ where the first row has entries $(\cos\theta \quad -\sin\theta)$, and each matrix of the second type is a matrix $S_\theta$ where the first row has entries $(\cos\theta \quad \sin\theta)$.

The first objective is to show that $B$ lies in the same arc component as a matrix with no summands of the second type. Express the matrix $B$ explicitly as a block sum as follows:

$$B = I_k \oplus \left( \bigoplus_{i=1}^{p} R_{\theta(i)} \right) \oplus \left( \bigoplus_{j=1}^{q} R_{\varphi(j)} \right) \oplus -I_\ell$$

Here $I_m$ denotes an $m \times m$ identity matrix. Consider the continuous curve in $\mathbf{SO}(n)$ defined by the formula

$$C_1(t) = I_k \oplus \left( \bigoplus_{i=1}^{p} R_{t\theta(i)} \right) \oplus I_{2q+\ell} \ .$$

It follows that $C_1(1) = B$ and $C_1(0) = B_1$ is a block sum matrix with no summands of the second type.

The next objective is to show that $B_1$ lies in the same arc component as a matrix with no summands of the third type. This requires more work than the previous construction, and the initial step is to show that one can find a matrix in the same arc component with at most one summand of the third type. The crucial idea is to show that if the block sum has $q \geq 2$ summands of the third type, then it lies in the same arc component as a matrix with $q-2$ block summands of the third type. In fact, the continuous curve joining the two matrices will itself be a block sum, with one $4 \times 4$ summand corresponding to the two summands that are removed and identity matrices corresponding to the remaining summands. Therefore the argument reduces to looking at a $4 \times 4$ orthogonal matrix that is a block sum of two $2 \times 2$ matrices of the third type, and the objective is to show that such a matrix lies in the same arc component as the identity matrix.

Given real numbers $\alpha$ and $\beta$, let $S_\alpha$ and $S_\beta$ be defined as above, and likewise for $R_\alpha$ and $R_\beta$. Then one has the multiplicative identity

$$S_\alpha \oplus S_\beta \quad = \quad \big(R_\alpha \oplus R_\beta\big) \cdot \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big) \ .$$

We have already shown that there is a continuous curve $C(t)$ in the orthogonal group such that $C(0) = I$ and $C(1) = R_\alpha \oplus R_\beta$. If we can construct a continuous curve $Q(t)$ such that $Q(0) = I$

and $Q(1) = \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big)$ . Then the matrix product $C(t)\,Q(t)$ will be a continuous curve in the orthogonal group whose value at 0 is the identity and whose value at 1 is $S_\alpha \oplus S_\beta$. But $R_\alpha \oplus R_\beta$ is a rotation by 180 degrees in the second and fourth coordinates and the identity on the first and third coordinates, so it is natural to look for a curve $Q(t)$ that is rotation through $180\,t$ degrees in the even coordinates and the identity on the odd ones. One can write down such a curve explicitly as follows:

$$Q(t) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t & 0 & -\sin \pi t \\ 0 & 0 & 1 & 0 \\ 0 & \sin \pi t & 0 & \cos \pi t \end{pmatrix}$$

Repeated use of this construction yields a matrix with no block summands of the first type and at most one summand of the second type. If there are no summands of the third type, we have reached the second objective, so assume that there is exactly one summand of the third type. Since our block sum has determinant 1, it follows that there is also at least one summand of the fourth type. We claim that the simplified matrix obtained thus far lies in the same arc component as a block sum matrix with no summands of the third type and one fewer summands of the fourth type. As in the previous step, everything reduces to showing that a $3 \times 3$ block sum $S_\alpha \oplus (-1)$ lies in the same arc component as the identity. In this case one has the multiplicative identity

$$S_\alpha \oplus (-1) \quad = \quad \big(R_\alpha \oplus 1\big) \oplus \Big(1 \oplus (-1) \oplus (-1)\Big)$$

and the required continuous curve is given by the formula

$$C(t) \quad = \quad \big(R_{t\alpha} \oplus 1\big) \oplus \Big((1 \oplus R_{t\pi}(-1)\Big) \ .$$

Thus far we have shown that every block sum matrix in $\mathbf{SO}(n)$ lies in the same arc component as a diagonal matrix (only summands of the first and last types). Let $I_s \oplus -I_\ell$ be this matrix. Since its determinant is 1, it follows that $\ell$ must be even, so write $\ell = 2k$. Then the diagonal matrix is a block sum of an identity matrix and $k$ copies of $R_\pi$, and one can use the continuous curve $D(t)$ that is a block sum of an identity matrix with $k$ copies of $R_{t\pi}$ to show that the diagonal matrix $I_s \oplus -I_\ell$ lies in the same arc component as the identity.

We have now shown that every block sum matrix with determinant 1 lies in the arc component of the identity. Suppose that $\Gamma(t)$ is a continuous curve whose value at 0 is the identity and whose value at 1 is the block sum. At the beginning of this proof we noted that for every orthogonal matrix $A$ we can find another orthogonal matrix $P$ such that $B = P\,A\,P^{-1}$ is is a block sum of the type discussed above; note that the determinant of $B$ is equal to 1 if the same is true for the determinant of $A$. If $\Gamma(t)$ is the curve joining the identity to $B$, then $P^{-1}\,\Gamma(t)\,P$ will be a continuous curve in $\mathbf{SO}(n)$ joining the identity to $A$.■

**4.** Show that $\mathbf{O}(n)$ has two connected components both homeomorphic to $\mathbf{SO}(n)$ for every $n$.

SOLUTION.

One can use the same sort of arguments that established topological product decompositions for $\mathbf{O}(2)$ and $\mathbf{SU}(n)$ to show that $\mathbf{O}(n)$ is homeomorphic to $\mathbf{SO}(n) \times \mathbf{O}(1)$. Since $\mathbf{SO}(n)$ is connected, this proves that $\mathbf{O}(n)$ is homeomorphic to a disjoint union of two copies of $\mathbf{SO}(n)$.■

**5.** Show that the inclusions of $\mathbf{O}(n)$ in $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{U}(n)$ in $\mathbf{GL}(n, \mathbb{C})$ determine 1–1 correspondences of path components. [*Hint:* Show that the Gram-Schmidt orthonormalization process expresses an arbitrary invertible matrix over the real numbers as a product $PQ$ where $P$ is upper triangular and $Q$ is orthogonal (real case) or unitary (complex case), and use this to show that every invertible matrix can be connected to an orthogonal or unitary matrix by a continuous curve.]

SOLUTION.

Let $A$ be an invertible $n \times n$ matrix over the real or complex numbers, and consider the meaning of the Gram-Schmidt process for matrices. Let $\mathbf{a}_j$ represent the $j^{\text{th}}$ column of $A$, and let $B$ be the orthogonal or unitary matrix whose columns are the vectors $\mathbf{b}_j$ obtained from the columns of $A$ by the Gram-Schmidt process. How are they related? The basic equations for defining the columns of $B$ in terms of the columns of $A$ have the form

$$\mathbf{b}_j = \sum_{k \leq j} c_{k,j} \mathbf{a}_k$$

where each $c_{j,j}$ is a positive real number, and if we set $c_{k,j} = 0$ for $k > j$ this means that $B = AC$ where $C$ is lower triangular with diagonal entries that are positive and real. We claim that there is a continuous curve $\Gamma(t)$ such that $\Gamma(0) = I$ and $\Gamma(1) = C$. Specifically, define this curve by the following formula:

$$\Gamma(t) \quad = \quad I \; + \; t\,(C - I)$$

By construction $\Gamma(t)$ is a lower triangular matrix whose diagonal entries are positive real numbers, and therefore this matrix is invertible. If we let $\Phi(t)$ be the matrix product $A\,\Gamma(t)$ then we have a continuous curve in the group of invertible matrices such that $\Phi(1) = A$ and $\Phi(0)$ is orthogonal or unitary. Therefore it follows that every invertible matrix is in the same arc component as an orthogonal or unitary matrix.

In the unitary case this proves the result, for the arcwise connectedness of $\mathbf{U}(n)$ and the previous argument imply that $\mathbf{GL}(n, \mathbb{C})$ is also arcwise connected. However, in the orthogonal case a little more work is needed. The determinant of a matrix in $\mathbf{GL}(n, \mathbb{R})$ is either positive or negative, so the preceding argument shows that an invertible real matrix lies in the same arc component as the matrices of $\mathbf{SO}(n)$ if its determinant is positive and in the other arc component of $\mathbf{SO}(n)$ if its determinant is negative. Therefore there are at most two arc components of $\mathbf{GL}(n, \mathbb{R})$, and the assertion about arc components will be true if we can show that $\mathbf{GL}(n, \mathbb{R})$ is not connected. To see this, note that the determinant function defines a continuous onto map from $\mathbf{GL}(n, \mathbb{R})$ to the disconnected space $\mathbb{R} - \{0\}$.■