Intersections of nested closed subsets

The following result is stated without proof in Section III.2 of the course notes:

PROPOSITION. (Nested Intersection Property). *Let X be a complete metric space, and let $\{ A_n \}$ be a nested sequence of nonempty closed subsets of X such that* $\lim_{n\to\infty} \text{diam}\,(A_n) = 0$. *Then $\cap_n\, A_n$ consists of one point.*

Since the given reference for the proof merely sends the reader to one of the exercises in Munkres, we shall give a full proof here.

Proof. First of all, we can find a subsequence $\{ B_k \}$ of $\{ A_n \}$ such that diam $(B_k) < 1/k$ for all k, and since $\cap_n\, A_n = \cap_k\, B_k$ it suffices to show that the latter consists of a single point. There are now two cases.

CASE 1. Suppose that some set B_k is finite; it follows that for each $q \geq k$ the set B_q is also finite. If we choose q such that $1/q$ is less than the minimum distance between two distinct points in B_k, then it follows that B_q consists of a single point, and hence $r \geq q$ implies that $B_r = B_q$ (since each set is nonempty). Therefore $\cap_k B_k = B_q$ is a single point.

CASE 2. Suppose that every set B_k is infinite. Then we can pick a sequence of points $\{b_k\}$ such that $b_k \in B_k$ and $b_k \neq b_i$ for $i < k$. We claim that $\{b_k\}$ is a Cauchy sequence. Given $\varepsilon > 0$, choose M such that $1/M < \varepsilon$. If $p, q \geq M$, then both b_p and b_q belong to B_M, so that $\mathbf{d}(b_p, b_q) \leq 1/M < \varepsilon$. Hence $\{b_k\}$ is a Cauchy sequence, and by completeness it has a limit b.

Suppose now that k is arbitrary. Since $b_i \in B_k$ for $i \geq k$ and B_k is closed, it follows that $b \in B_k$, and since k is arbitrary this implies that $b \in \cap_k\, B_k$.

One can now prove that the intersection consists of exactly one point by means of the argument with appears after the statement of the Nested Intersection Property in the course notes.■