Background information on projective spaces

At some points in Units V and VI (and the accompanying exercises) we discuss a topological space called the **real projective plane**. The following online directory contains a fairly complete set of notes describing the geometry of such objects at the level of an upper level undergraduate course:

`http://math.ucr.edu/~res/progeom`

Projective spaces play an important role in many branches of mathematics, and in particular they are used extensively in a wide range of graduate level mathematics courses.

Projective spaces and compactifications

Some basic topological properties of the real projective plane are developed in Additional Exercises 3 and 4 for Section V.1; both of these generalize directly to objects known as *real projective n-spaces* $\mathbb{RP}^n$ for all $n \geq 2$; more precisely, the latter may be defined as the quotient of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$ by the equivalence relation which identifies two vectors if and only if they are nonzero multiples of each other.

These spaces may be viewed as a compactification of $\mathbb{R}^n$ which can be obtained by adding points at infinity for each family of parallel lines. A discussion of this from first principles is contained in the lecture notes for Mathematics 205C

`http://math.ucr.edu/~res/math205C/lectnotes.*`

specifically in Subsection V.1.1 on pages 165–172. Some of the material covered there is beyond the scope of this course (particularly everything about smooth structures), but the main discussion should be understandable to a student who is near the end of Mathematics 205A. The statement that one obtains a compact Hausdorff space appears on page 169. Note that the discussion is formulated to produce spaces $\mathbb{FP}^n$ extending $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

The discussion in the Mathematics 205C lecture notes does not explicitly show that the image of the standard (continuous open) inclusion h of $\mathbb{F}^n$ in $\mathbb{FP}^n$ given by

$$h(x_1, \cdots, x_n) \quad = \quad \text{Span of } (x_1, \cdots, x_n, 1)$$

is dense, but this can be verified very quickly as follows:

CLAIM. *Given a point J not in the image of h and an open neighborhood W containing J, there is a point $h(x) \in W$.*

Construction. By definition, $\mathbb{FP}^n$ is a quotient space of $\mathbb{F}^{n+1} - \{0\}$ such that the equivalence class of a point is all nonzero scalar multiples of that point and the image of h is the set of all equivalence classes whose representatives have a nonzero last coordinate. Therefore, if J is not in the image of h then the last coordinate for any vector v representing J must be zero. Suppose that we take an open neighborhood U of J, and we consider its (open) inverse image U^* in $\mathbb{F}^{n+1} - \{0\}$. Then U^* clearly contains a vector w whose last coordinate is nonzero, and if Y is the equivalence class of w, then Y lies in the image of h and also $y \in U$. Therefore the set $h\,[\mathbb{F}^n]$ is dense in $\mathbb{FP}^n$.■