Partitions of Unity and a Metrization Theorem of Smirnov

Paracompactness and partitions of unity both play important roles in the applications of point set topology to other branches of mathematics, and for that reason both appear frequently in elementary topology courses. One difficulty in presenting these topics is a shortage of examples that illustrate the uses of partitions of unity but do not require concepts beyond the scope of an elementary course (for example, vector bundles and riemannian metrics). A few examples of this sort are given in the texts by Dugundji [D] and Munkres [M]. The purpose of this article is to show that the following standard result can also be used as an example.

Theorem. *Let X be a space that is paracompact Hausdorff, and assume that every point of X has an open neighborhood that is metrizable. Then X is metrizable.*

This result follows quickly from the Nagata-Smirnov characterization of metrizable spaces as T_3 spaces with σ-locally finite bases. However, the uses of partitions of unity suggest a more direct approach; namely, the construction of a global metric for the space from the local metrics on the neighborhoods and a partition of unity. Our objective is to provide a mathematical justification for this idea. A direct proof of Smirnov's result along these lines has several useful features. First, it accurately illustrates the ways in which partitions of unity are used in more advanced mathematics; furthermore, an approach of this sort allows the inclusion of Smirnov's theorem even if the general metrizability theorem of Nagata and Smirnov is not covered in a course. Here is a published reference for the main ideas:

I. Colojoară, *Rev. Roum. math. pures et appl.* **10** (1965), pp. 291–296 & 1051–1052.

The following observation will be used in the proof of the main theorem:

Lemma. *If M is a metric space, then M is homeomorphic to a subset of a normed real vector space.*

Proof of Lemma. Since every metric space is isometric to a subset of a complete metric space, it suffices to consider the case where M is complete. Furthermore, we may assume that M is bounded with diameter ≤ 1 by replacing the original metric d with the minimum of d and 1. Let $\mathbf{BC}(M)$ denote the space of bounded real valued continuous functions on M with the usual supremum norm, and define $f : M \to \mathbf{BC}(M)$ by $f_x(t) := d(x,t)$. Since $|f_x - f_y| \leq |x - y|$, the map $x \to f_x$ is uniformly continuous. It is also one to one, for $f_x = f_y$ implies $f_x(x) = f_y(x)$, and the latter implies $x = y$. Finally, we claim f is closed. Let A be closed in M, and let $\{x_n\}$ be a sequence in A such that f_{x_n} converges to some function g; we need to show that $g = f_x$ for some $x \in A$. But $\{f_{x_n}\}$ is a Cauchy sequence because it converges; given $\varepsilon > 0$ choose N such that $m, n \geq N$ implies $|f_{x_m} - f_{x_n}| < \varepsilon$. Since the value of the left hand side is just $d(x_m, x_n)$, it follows that $\{x_n\}$ is also a Cauchy sequence. Since A is closed in M and M is complete, it follows that A is also complete, so that $\{x_n\}$ converges to some point $x \in A$. By continuity we must have $g = f_x$.

Proof of Smirnov's Theorem. By assumption there is an open covering of X for which every subset is metrizable, and by paracompactness there is also a locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of this type. Let $\{\varphi_\alpha\}$ be a partition of unity subordinate to $\mathcal{U}$. Since each subset U_α is metrizable, the lemma implies the existence of normed vector spaces L_α and maps $f_\alpha : U_\alpha \to L_\alpha$ that are homeomorphisms onto their images.

Let $\sum_\alpha (L_\alpha \times \mathbb{R})$ be the direct sum (= all tuples $\{y_\alpha\}$ such that $y_\alpha = 0$ for at most finitely many α) with the norm given by the sum of the norms of the factors. Define a continuous map

$$F : X \to \sum_\alpha (L_\alpha \times \mathbb{R})$$

by the formula

$$F(x)_{\alpha-\text{coord.}} \quad = \quad \big(\varphi_\alpha(x) f_\alpha(x),\, \varphi_\alpha(x)\big).$$

By local finiteness, for each $x \in X$ only finitely many $\varphi_\alpha(x) \neq 0$, and therefore the formula defines a map into the direct sum. The continuity of F follows directly from the continuity of the maps f_α and φ_α.

We claim that F is a homeomorphism onto its image. First of all, we shall verify that F is one to one. Suppose $F(x) = F(y)$; then by definition $\varphi_\alpha(x) = \varphi_\alpha(y)$ for all α. Choose α_0 so that $\varphi_{\alpha_0} > 0$ at both points. Then $x, y \in W_{\alpha_0} = \varphi_{\alpha_0}^{-1}(0,1]$. Since the projections of $F(x)$ and $F(y)$ onto L_{α_0} are given by $\varphi_{\alpha_0}(x) f_{\alpha_0}(x)$ and $\varphi_{\alpha_0}(y) f_{\alpha_0}(y)$ respectively, it follows that $f_{\alpha_0}(x) = f_{\alpha_0}(y)$. Since F_{α_0} is one to one this implies $x = y$.

We next claim that the following hold:

(*i*) F maps each set W_α as above homeomorphically onto its image.
(*ii*) $F(W_\alpha)$ is open in $F(X)$.

In the final paragraph we shall show that F is a homeomorphism onto its image using (*i*) and (*ii*).

To prove (*i*), let $h_\alpha : W_\alpha \to F(W_\alpha)$ be the 1–1 continuous map induced by F, let π_α be the projection of the sum onto the α factor, and let p_α and s_α be the projections of $\pi_\alpha F | W_\alpha$ to L_α and $\mathbb{R}$ respectively. By construction $s_\alpha > 0$ everywhere, and by construction it follows that $f_\alpha(x) = s_\alpha(h_\alpha(x))^{-1} p_\alpha(h_\alpha(x))$ for all $x \in W_\alpha$. Hence there is a continuous map $q_\alpha : F(W_\alpha) \to f_\alpha(W_\alpha)$ such that $q_\alpha h_\alpha = f_\alpha$; it follows that h_α is a homeomorphism, for if $g_\alpha : f_\alpha(W_\alpha) \to W_\alpha$ is an inverse to f_α, then $g_\alpha q_\alpha$ is an inverse to h_α. To prove (*ii*), let t_α be the composite projection $\pi_{\mathbb{R}} \pi_\alpha$ and note that $F(W_\alpha) = F(X) \cap t_\alpha^{-1}(\mathbb{R} - \{0\})$.

Using (*i*) and (*ii*), we conclude the proof as follows. Let V be open in X. Then by (*i*) for each α set $F(V \cap W_\alpha)$ is open in $F(W_\alpha)$. But by (*ii*) each $F(W_\alpha)$ is open in $F(X)$, and therefore each set $F(V \cap W_\alpha)$ is open in $F(X)$. Since $\sum \varphi_\alpha = 1$ and $\varphi_\alpha \geq 0$, for each $x \in X$ there is some $\alpha(x)$ such that $\varphi_{\alpha(x)} > 0$; therefore the sets W_α define an open covering of X. It follows that $F(V)$ is the union of the sets $F(V \cap W_\alpha)$ and consequently is open in $F(X)$.

Final remark. Metrizable spaces are paracompact by a well-known theorem of A. H. Stone, and in some sense Smirnov's theorem is a converse to Stone's result.

References

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