FACE IDENTITIES

$0 \leq i < j$

$\partial_i \circ \partial_j$ = remove vertex v_j, then remove v_i.

One gets the same result from the composite

$$\partial_{j-1} \circ \partial_i .$$

$$d_{n-1} \circ d_n = \left(\sum (-1)^i \partial_i\right) \circ \left(\sum (-1)^j \partial_j\right) =$$

$$\sum_{i,j} (-1)^{i+j} \partial_i \circ \partial_j = \sum_{i<j} (-1)^{i+j} \partial_i \partial_j +$$

$\sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j$. Apply the Face Identities to the first sum, getting

$$\sum_{i<j} (-1)^{i+j} \partial_{j-1} \partial_i + \sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j .$$

Do a change of variables for the first term, with $u = j-1$ $v = i$. Then we get

$$\sum_{u \geq v} (-1)^{u+v-1} \partial_u \partial_v + \sum_{i \geq j} (-1)^{i+j} \partial_i \partial_j .$$

Notice that the first and second terms cancel each other, so that

$$\boxed{d_{n-1} \circ d_n = 0.}$$