# Complex logarithmic functions

One inconvenient fact about logarithms is that there is good way to extend the usual definition over the positive real numbers to the entire set of nonzero complex numbers. However, complex logarithmic functions can be defined locally because the complex derivative of the complex exponential function $\exp(x + iy) = e^x(\cos y + i \sin y)$ is always nonzero, the Inverse Function Theorem in complex analysis (see Lang, *Complex Analysis*, Theorem 6.1.(c), p. 76). For many purposes it is useful to know that complex logarithmic functions exist on open sets which are larger than a sufficiently small open neighborhood of some point, and we can use covering space theory to obtain a reasonably strong conclusion. Before proceeding, we shall give a formal definition.

**Definition.** Let $U$ be an open connected subset of $\mathbb{C} - \{0\}$, where $\mathbb{C}$ denotes the complex plane. A continuous function $L : U \to \mathbb{C} - \{0\}$ is said to be **a branch of** $\log z$ if for all $z \in U$ we have $z = \exp \circ L(z)$.

One can use the previously cited Inverse Function Theorem to conclude that if a branch of $\log z$ exists on an open subset $U$ then it is automatically complex analytic (since this is true locally by the Inverse Function Theorem).

The first step is to prove an elementary property of the complex exponential function.

**PROPOSITION.** *The exponential mapping* $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$ *is a covering space projection.*

**Sketch of proof.** Every nonzero point $w_0$ in the complex plane has the form $\exp(x_0 + iy_0)$ for some $x_0$ and $y_0$. Let $V \subset \mathbb{C} \cong \mathbb{R}^2$ be the set of all points $(x, y)$ such that $y - y_0 \neq \pi n$ for some odd integer $n$, so that $W$ is a union of pairwise disjoint horizontal strips of height $2\pi$. Each of these strips maps homeomorphically to the set of all nonzero complex numbers $z$ such that

$$\frac{z}{|z|} \quad \neq \quad -\frac{w_0}{|w_0|}$$

and this implies that the open subset defined by the displayed inequality is evenly covered with respect to the exponential map.∎

We can now state and prove the main result:

**THEOREM.** *Suppose that* $U$ *is an open simply connected subset of* $\mathbb{C} - \{0\}$. *Then it is possible to define a branch of* $\log z$ *on* $U$.

It is worthwhile to note that the imaginary part of such a function is not necessarily bounded between two real numbers such as $t$ and $t + 2N\pi$, where $N$ is some positive integer. Specifically, if $V \subset \mathbb{C} = \mathbb{R}^2$ is the open set defined by $x, y > 0$ and $2\pi - x < y < x$ — which is the complement of the Archimedean spiral with polar coordinate equation $r = \theta$ — then $V$ is convex (hence simply connected) and the exponential map sends $V$ homeomorphically onto its image, and in this case the branch of $\log z$ simply corresponds to the inverse function. (It may be worthwhile to draw a picture and visualize this.)

**Proof.** This is merely an application of the preceding proposition Lifting Principle because the map of fundamental groups $\pi_1(U) \to \pi_1(\mathbb{C} - \{0\})$ is trivial whenever $U$ is simply connected.∎

Note that the Lifting Principle also implies that a branch of $\log z$ exists if and only if the map of fundamental groups is trivial.