Another Exercise on Plane Separation

Let C_1 and C_2 be disjoint simple closed curves in S^2, and let $X = C_1 \cup C_2$. Then $S^2 - X$ has three components U_1, U_2, V such that

(i) For $i = 1$ or 2, $H_*(U_i) \cong H_*(\text{point})$ and the boundary of U_i is C_i.

(ii) $H_*(V) \cong H_*(S^1)$ and the boundary of V is $C_1 \cup C_2$. Furthermore, if $p_i \in U_i$ then $H_*(V) \longrightarrow H_*(S^2 - \{p_1, p_2\})$ is an isomorphism.

In fact, fundamental theorems of plane topology imply that $\overline{U_i} \cong D^2$ and $\overline{V} \cong S^1 \times [0,1]$. The first of these is the Schönflies Theorem,* and the second is the "Annulus Conjecture"* (it's really a theorem).

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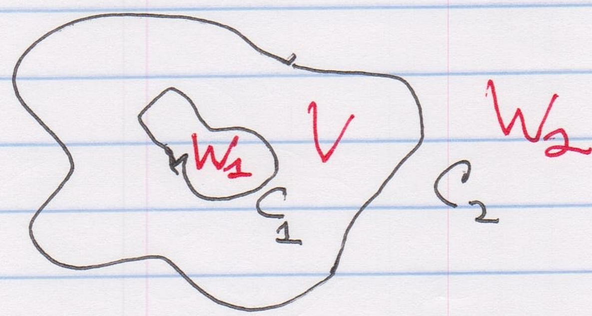
* In the plane. Both fail to be true for $C_i \cong S^{n-1} \subseteq S^n$.

There are many similarities between this solution and the one in solutions05b.pdf.



SOLUTION Start with the Jordan Curve Thm.

$i = 1, 2 \Rightarrow$ $S^2 - C_i = W_i \cup U_i$, where W_i, U_i are disjoint open subsets and

$$\text{Bdy } W_i = C_i = \text{Bdy } U_i.$$

Since C_1 is connected and $C_1 \subseteq S^2 - C_2$, either $C_1 \subseteq U_2$ or $C_1 \subseteq W_2$. Without loss of generality, we might as well assume $C_1 \subseteq W_2$. Likewise

——— (DITTO) ——— $C_2 \subseteq W_1$.

Since $S^2 - X \subseteq S^2 - C_i$ and $U_1 \cap C_2 = \phi$ and U_1 is a maximal connected subset of $S^2 - C_1$, it follows that U_1 is a component of $S^2 - X$. Similar considerations apply to U_2.

By the proof of the Jordan Curve Thm.,

$$H_*(U_i) \cong H_*(pt) \cong H_*(W_i).$$

SECOND STEP Claim $S^2 - X$ has 3 components.

Write $S^2 = (S^2 - C_1) \cup (S^2 - C_2)$, so that

$$X = (S^2 - C_1) \cap (S^2 - C_2),$$ and we have a Mayer-Vietoris sequence as follows:

$$\begin{array}{ccccccccc} H_1(S^2) & \longrightarrow & H_0(S^2 - X) & \longrightarrow & \begin{matrix} H_0(S^2 - C_1) \\ \oplus \\ H_0(S^2 - C_2) \end{matrix} & \longrightarrow & H_0(S^2) & \longrightarrow & 0 \\ \| & & \| & & \| & & & & \\ 0 & \longrightarrow & ? & \longrightarrow & \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \xrightarrow{\text{ADD}} & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

It follows that $H_0(S^2 - X) \cong \mathbb{Z}^3$.

We have already seen that U_1 & U_2 are components of $S^2 - X$, and $U_1 \neq U_2$ because $\mathrm{Bdy}\, U_1 = C_1 \neq C_2 = \mathrm{Bdy}\, U_2$. Therefore there is one additional component that we shall call V. — We need to verify the assertions in (iii).

We have

$$S^2 = U_1 \cup C_1 \cup \overbrace{U_2 \cup C_2}^{= W_1} \cup V.$$

We first prove $H_*(V) \cong H_*(S^1)$. It will suffice to prove $H_i(S^2 - X) \cong H_i(S^1)$ if $i > 0$.

Consider the Mayer-Vietoris sequence $(i>0)$

$$\begin{matrix} H_{i+1}(S^2 - C_1) \\ \oplus \\ H_{i+1}(S^2 - C_2) \\ \| \\ 0 \\ (i>0) \end{matrix} \to H_{i+1}(S^2) \xrightarrow{\Delta} H_i(S^2 - X) \to \begin{matrix} H_i(S^2 - C_1) \\ \oplus \\ H_i(S^2 - C_2) \\ \| \\ 0 \\ (i>0) \end{matrix}$$

Hence Δ is an isomorphism.

Next, we prove $\operatorname{Bdy} V = C_1 \cup C_2$.

Now $\operatorname{Bdy} V \subseteq C_1 \cup C_2$ because $V \cup C_1 \cup C_2 = S^2 - (U_1 \cup U_2)$ is closed, so $\overline{V} \subseteq V \cup C_1 \cup C_2$ and $\operatorname{Bdy} V = \overline{V} - V \subseteq C_1 \cup C_2$.

Conversely, suppose $z \in C_1 \cup C_2$. Say $z \in C_1$ (the other case follows similarly).

limit point set

We know $z \in L(W_1)$, so if N is an open neighborhood of z we have $(N - \{z\}) \cap W_1 \neq \emptyset$. In fact, we can say more. Since $z \notin \underbrace{U_2 \cup C_2}_{= F_2} = S^2 - W_2$ and the latter is closed, we know that $N' = N - F_2$ is also an open neighborhood of z, so that $N' - \{z\} \cap W_1 \neq \emptyset$. But $(N' - \{z\}) \cap W_1 = (N - \{z\}) \cap V$, so we have that the latter is nonempty and hence $z \in L(V)$. Hence $C_1 \subseteq L(V)$ and similarly $C_2 \subseteq L(V)$, so that $C_1 \cup C_2 \subseteq \overline{V} - V = \mathrm{Bdy}\, V$.

At this point, it only remains to prove that the map induced by inclusion

$$H_*(V) \longrightarrow H_*(S^2 - \{p_1, p_2\})$$

is an isomorphism.

Let $p_1 \in U_1$, $p_2 \in U_2$, so that $\begin{cases} p_1 \in W_2 \\ p_2 \in W_1 \end{cases}$.

Let $W_1' = W_1 - \{p_2\}$, $W_2' = W_2 - \{p_1\}$.

We then have $S^2 = W_1 \cup W_2$, $V = W_1 \cap W_2$,

$S^2 - \{p_1, p_2\} = W_1' \cup W_2'$, $V = W_1' \cap W_2'$.

Look at the associated diagram for Mayer-Vietoris sequences:

$$\begin{tikzcd} H_2(S^2 - \{p_1,p_2\}) \arrow[r, equal] \arrow[d] & H_1(V) \arrow[r] \arrow[d, "="] & H_1(W_1') \oplus H_1(W_2') \arrow[r, "J"] \arrow[d] & H_1(S^2 - \text{pts}) \arrow[d] \\ H_2(S^2) \arrow[r, "\Delta", "\cong"'] & H_1(V) \arrow[r] & H_1(W_1) \oplus H_1(W_2) \arrow[r] & H_1(S^2) \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "="] & ?? \arrow[r] & \mathbb{Z} \arrow[r, "\Delta_0"] & {} \\ \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

$\Delta_0 = 0$ because V is connected

(look at the maps in H_0, which must be 1-1)

Hence J is onto.

It follows that $H_1(W_1') \cong H_1(W_2') \cong \mathbb{Z}$.

Also, if $\begin{Bmatrix} K_1 \subseteq W_1' \\ K_2 \subseteq W_2' \end{Bmatrix}$ is a small circle centered

at $\begin{Bmatrix} p_2 \\ p_1 \end{Bmatrix}$ then $\underset{\mathbb{Z}}{H_1(K_i)} \to H_1(W_i') \to \underset{\mathbb{Z}}{H_1(S^2 - pts)}$

is an isomorphism. Hence each map

$H_1(W_i') \longrightarrow H_1(S^2 - pts)$ is also an

isomorphism and image $H_1(V) \longrightarrow$

$H_1(W_1') \oplus H_1(W_2') \cong \mathbb{Z} \oplus \mathbb{Z}$ must have

the form (g_1, g_2) where g_i generates

$H_1(W_i')$. A diagram chace now shows

that $H_1(V) \longrightarrow H_1(W_i') \longrightarrow H_1(S^2 - pts)$

must be an isomorphism.