# Addendum to `concentric.pdf`

On the first page of the cited document, the following results were mentioned:

**Schönflies Theorem in 2 dimensions.** *Let $C \subset S^2$ be homeomorphic to $S^1$, and let $U$ be one of the two components of $S^2 - C$ (recall that the latter has 2 components by the Jordan-Brouwer Separation Theorem). Then every homeomorphism $S^1 \to C$ extends to a homeomorphism from $D^2$ to $C \cup U$.*

**Annulus Theorem in 2 dimensions.** *Let $C_1, C_2 \subset S^2$ be disjoint subsets, each of which is homeomorphic to $S^1$, and let $V$ be the unique component of $S^2 - (C_1 \cup C_2)$ whose homology groups are isomorphic to those of $S^1$. Then $C_1 \cup V \cup C_2$ is homeomorphic to $S^1 \times [0,1]$ such that $C_i$ corresponds to $S^1 \times \{i-1\}$ for $i = 1, 2$.*

The *Wikipedia* articles on the Schönflies Problem and Annulus Theorem give accessible (and reasonably accurate) descriptions of the background for these and related results, and they also provide several references for proofs. We shall limit this note to discussing and clarifying some points which appear in these articles.

Counterexamples in higher dimensions. The article on the Schönflies Problem mentions the Alexander Horned Sphere as an example of a subset in $S^3$ which is homeomorphic to $S^2$ but whose complementary regions are "not homeomorphic to the inside and outside of a normal sphere." This statement may be misleading, for in the standard construction of the Alexander Horned Sphere the inside region is homeomorphic to the inside of a standardly embedded 2-sphere but the outside is not. However, a result of R. H. Bing (Annals of Mathematics **56** (1952), pp. 354–362) implies that there is a topological embedding of $S^2$ in $S^3$ such that neither the inside nor the outside region is homeomorphic to the inside or outside of the standardly embedded 2-sphere.

Sufficient conditions for theorems in higher dimensions. As noted in the *Wikipedia* articles, both the Schönflies and Annulus Theorem have generalizations to higher dimensions if we restrict to embeddings of $S^{n-1}$ in $S^n$ — or of two disjoint copies of $S^{n-1}$ in $S^n$ — which are sufficiently well behaved. In fact there is a fairly weak sufficiency hypothesis which is easily stated. Namely, both result generalize if we assume that the embedded copy or copies of $S^{n-1}$ are open neighborhoods which are homeomorphic to $S^{n-1} \times (-1, 1)$ such that the embedded sphere corresponds to $S^{n-1} \times \{0\}$ (*i.e.*, each sphere has a bi-collar neighborhood).

The condition in the preceding sentence is automatically satisfied if each sphere is smoothly embedded (in the sense of Mathematics 205C), and it is also satisfied if the embedding is piecewise smooth in an appropriate sense. Examples of the latter include the usual realizations of 2-dimensional polyhedra in $\mathbb{R}^3$ such as cubes, pyramids (including simplices), prisms, regular and semi-regular polyhedra, and so on, provided the polyhedron under consideration is homeomorphic to $S^2$.