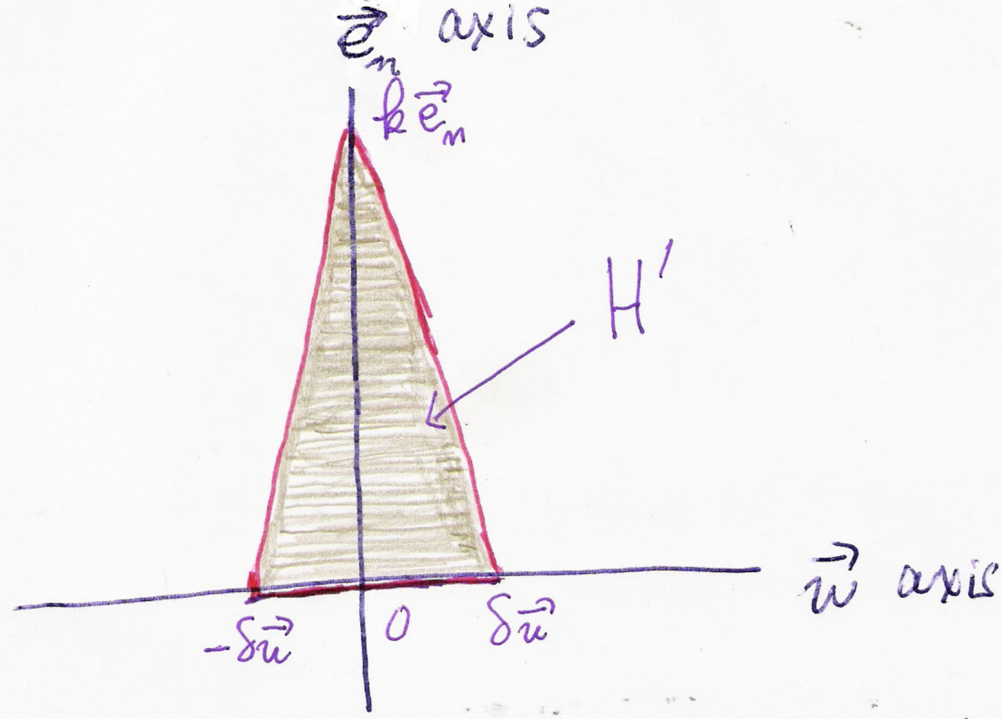
Planar section of the solid cone

$$x_n \geq 0, \quad \frac{x_n}{k} \leq 1 - \frac{1}{\delta}\sqrt{\sum_{i=1}^{n-1} x_i^2}$$



[We can think of this as the set one gets by setting $x_2 = \ldots x_{n-1} = 0$.] typical example

H' is convex and contains $k\vec{e}_n$ plus the line segment of all $t\delta\vec{u}$, where $|t| \leq 1$. Hence H' contains the solid triangular region with vertices $k\vec{e}_n$ and $\pm\delta\vec{u}$.

Here is a justification for the preceding sentence. A typical point in the right hand solid triangle has the form

$$p\,\vec{e}_n + q\,\vec{u}$$

where $p, q \geq 0$ and $\frac{p}{k} + \frac{q}{\delta} \leq 1$. We can write this as $\frac{p}{k}(k\vec{e}_n) + \frac{q}{\delta}(\delta\vec{u}_n) +$

NOTE THIS LIES IN $[0,1]$.

$$\left(1 - \left(\tfrac{p}{k}\right) - \left(\tfrac{q}{\delta}\right)\right)\cdot\vec{0}.$$

Now a convex set is closed under convex combinations

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$\sum t_i x_i$, where $t_i \geq 0$ and $\sum t_i = 1$, so every point as above must lie in the convex set H'. Similar considerations apply to the left hand solid triangle if we replace $\vec{u}$ by $-\vec{u}$ throughout the discussion. ∎

Lemma Suppose $K \subseteq \mathbb{R}^n$ is convex. If $x_0, \ldots, x_m \in K$ and $t_0, \ldots, t_m \geq 0$ are such that $\sum t_i = 1$, then $\sum t_i x_i \in K$.

Proof. True by definition if $m = 1$; assume it's ~~true~~ known for $m \leq k-1$. Let $s = \sum_{i=0}^{m-1} t_i$. If $s = 0$ then $t_i = 0$ for $i < m$ and $\sum t_i x_i = x_m$. Suppose $s > 0$. Then by induction we have $\sum_{i=0}^{m-1} \frac{t_i}{s} x_i = y \in K$. Note that $\sum_{i=0}^{m-1} \frac{t_i}{s} = 1$. But now we have $\sum_{i=0}^{m} t_i x_i = s y + t_m x_m$, and since $y \in K$, $s, t_m \geq 0$ and $s + t_m = 1$ it follows that

$$\sum_{i=0}^{m} t_i x_i \in K. \blacksquare$$

One More Thing

We shall verify the assertion at the end of convexbodies.pdf about the convexity of the set defined by

$$0 \leq \frac{x_n}{k} \leq 1 - \frac{1}{\delta}\sqrt{\sum_{i=0}^{n-1} x_i^2}.$$

Note $x_n \geq 0 \iff \frac{x_n}{k} \geq 0$

Suppose $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ lie in this set. Let $P : \mathbb{R}^n \to \mathbb{R}^{n-1}$ forget the last coordinate, so that

$$0 \leq \frac{x_n}{k} \leq 1 - \frac{1}{\delta}|Px| \quad \text{and similarly for } y.$$

Let $0 \leq t \leq 1$ and let $z = tx + (1-t)y$; write $z = (z_1, \ldots, z_n)$. We need to show z lies in the given set. First of all, we have

$$0 \leq \frac{tx_n}{k} + \frac{(1-t)y_n}{k} = \frac{z_n}{k}. \text{ Next,}$$

$$1 - \frac{1}{\delta}|Pz| = 1 - \frac{1}{\delta}|P(tx + (1-t)y)| = 1 - \frac{1}{\delta} \cdot$$ (continued on next line)

$$|tPx + (1-t)Py| \geq 1 - \frac{1}{\delta}\left(t|Px| + (1-t)|Py|\right) =$$

P is linear

Triangle Inequality

$$t\left(1 - \frac{1}{\delta}|Px|\right) + (1-t)\left(1 - \frac{1}{\delta}|Py|\right) \geq$$

HYPOTHESIS ON x, y

$$t\frac{x_n}{k} + (1-t)\frac{y_n}{k} = \frac{z_n}{k} \quad \blacksquare$$