Homology groups of disks with holes

THEOREM. *Let $p_1, \cdots, p_k\}$ be a sequence of distinct points in the interior unit disk D^n where $n \geq 2$, and suppose that for all j the sets $E_j \subset \mathrm{Int}\,\mathrm{D^n}$ are closed, pairwise disjoint subdisks. Let S_j denote the boundary of E_j, and let $F = D^n - \cup_j \mathrm{Int}E_j$. Then the following hold:*

(i) The homology groups $H_q(F)$ are zero if $q \neq 0, n-1$, and $H_0(F) \cong \mathbb{Z}$.

(ii) In the remaining dimension we have $H_{n-1}(F) \cong \mathbb{Z}^k$, and the inclusion induced mappings $H_{n-1}(S_j) \to H_{n-1}(F)$ send generators of the domains into a set of free generators for the codomain.

(iii) If we define standard generators for $H_{n-1}(S_j)$ by taking the images of the standard generator for $H_{n-1}(S^{n-1})$ under the canonial homeomorphims $S^{n-1} \to S_j$, then the image of the geerator for $H_{n-1}(S^{n-1})$ in $H_{n-1}(F)$ is equal to the sums of the standard free generators for $H_{n-1}(F)$.

Geometrically, F is a disk with k holes; a picture of one example is included on the last page of this document. The standard homeomorphisms $S^{n-1} \to S_j$ arise from the homeomorphisms from $\mathbb{R}^n$ to itself which send $v \in \mathbb{R}^n$ to $(r_j \cdot v) + p_j$, where r_j is the radius of E_j.

Proof of the theorem

The first step is to replace F by an open set.

CLAIM 1. *The closed set F is a strong deformation retract of $\mathbb{R}^n - \{p_1, \cdots, p_k\}$.*

Proof of Claim 1. Observe that $\mathbb{R}^n - \{p_1, \cdots, p_k\}$ is the union of F with the punctured closed disks $E_j - \{p_j\}$, and the intersection is $\cup_j S_j$. Therefore it is enough to show that for each j the sphere S_j is a strong deformation retract of $E_j - \{p_j\}$. We can construct the retractions $E_j - \{p_j\} \to S_j$ by the standard formula

$$\rho_j(v) \quad = \quad p_j \; + \; \frac{r_j}{|v - p_j|} \cdot (v - p_j)$$

for pushing points into the boundary radially, and if $\varphi_j : S_j \to E_j - \{p_j\}$ is the inclusion mapping then $\varphi \circ \rho_j$ is homotopic to the identity by a straight line homotopy.■

CLAIM 2. *If $D \subset \mathbb{R}^n$ is a closed metric disk, then $(\mathbb{R}^n)^\bullet - D$ is homeomorphic to the interior of D^n.*

Proof of Claim 2. The first step is to reduce the proof to the familiar case where D is the unit disk $D^n \subset \mathbb{R}^n$.

If the result is true when $D = D^n$, then it is also true for every closed disk E centered at the origin, for the homeomorphism $M_r : \mathbb{R}^n \to \mathbb{R}^n$ defined by $v \to r \cdot v$ (where $v > 0$) sends D^n to the disk of radius r, and for general reasons it extends to a homeomorphism of the one point compactification $(\mathbb{R}^n)^\bullet$. If r is the radius of E, then this homeomorphism sends D^n to E and hence also sends the complement of D^n homeomorphically to the complement of E.

Next, if the result is true for every closed disk E centered at the origin, then it is true for all closed metric disks D, for if p is the center of D, then $T(v) = v + p$ is a homeomorphism of $\mathbb{R}^n$ to itself and hence extends to a homeomorphism from $(\mathbb{R}^n)^\bullet$ to itself. If we choose E to have the same radius as D, then this homeomorphism maps D to E and hence also maps the complement of D to the complement of E.

Finally, we have to show the result is true for D^n. Let $G \subset \mathbb{R}^n$ be the set of all vectors v such that $|v| \geq 1$. We claim that G is homeomorphic to $D^n - \{\mathbf{0}\}$; an explicit homeomorphism is given by sending v to $|v|^{-2} \cdot v$, so that the image of v points in the same direction but has length $|v|^{-1}$. If we extend this homeomorphism to one point compactifications and note that the one point compactification of $D^n - \{\mathbf{0}\}$ is homeomorphic to D^n, we obtain a homeomorphism from $G^\bullet$ to D^n such that the unit sphere is sent to itself. Taking complements of the unit sphere, we see that $G^\bullet - S^{n-1} \cong (\mathbb{R}^n)^\bullet - D^n$ is homeomorphic to the interior of D^n.■

STEP 3. *Computation of $H_*(\mathbb{R}^n - \{p_1, \cdots, p_k\})$ for $n \geq 2$.*

Let U_j denote the interior of the disk E_j. Then by excision we have

$$H_*(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\}) \;\cong\; H_*(\cup_j U_j, \cup_j U_j - \{p_j\}) \;\cong\; \bigoplus_j H_*(U_j, U_j - \{p_j\})$$

where the second isomorphism holds because the homology of a space splits into the direct sum of the homology groups of its arc components. These relative groups are $\mathbb{Z}$ in dimension n and zero otherwise, and since $n \geq 2$ the result in this case follows because the long exact homology sequence of $(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\})$ yields isomorphisms from $H_{q+1}(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\})$ to $H_q(\mathbb{R}^n - \{p_1, \cdots, p_k\})$ if $q > 0$ and from H_0 of the latter to $H_0(\mathbb{R}^n) \cong \mathbb{Z}$.■

Before proceeding to the final step, we shall discuss the construction of canonical generators in more detail. For our purposes it will suffice to begin by taking a canonical generator for $H_{n-1}(bbR^n, \mathbb{R}^n = \{\mathbf{0}\}) \cong \mathbb{Z}$; there are ways of choosing such generators for all values of n canonically, but we shall not try to explain how this can be done.

Consider the following commutative diagram, in which $p \in \operatorname{Int} D^n$ and we identify S^n with the one point compactification of $\mathbb{R}^n$:

$$\begin{array}{ccccc}
H_{q-1}(S^n - \{p\}) & \longleftarrow & H_{q-1}(S^n - D^n) & \longrightarrow & H_{q-1}(S^n - \{\mathbf{0}\}) \\
\uparrow & & \uparrow & & \uparrow \\
H_q(S^n, S^n - \{p\}) & \longleftarrow & H_q(S^n, S^n - D^n) & \longrightarrow & H_q(S^n, S^n - \{\mathbf{0}\}) \\
\uparrow & & \uparrow & & \uparrow \\
H_q(\mathbb{R}^n, \mathbb{R}^n - \{p\}) & \longleftarrow & H_q(\mathbb{R}^n, \mathbb{R}^n - D^n) & \longrightarrow & H_q(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\})
\end{array}$$

By Claim 2 we know that $S^n - D^n$ is contractible, and we also know that $S^n - \{p\}$ is contractible for all p in the interior of D^n by the symmetry properties of S^n and the fact that $S^n - \{v\} \cong \mathbb{R}^n$ if v is the point at infinity. Therefore the homomorphisms in the first row of the diagram are isomorphisms. Next, the vertical arrows from the second row to the first are the boundary homomorphisms in long exact sequences of pairs, and therefore a Five Lemma argument shows that the homomorphisms in the second row are also isomorphisms. Finally, the vertical arrows from the third row to the second are excision isomorphisms, and therefore the homomorphisms in the third row are also isomorphisms. We can then use the third row to define a canonical generator for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{p\})$ by taking the class corresponding to the chosen generator for $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\})$.

STEP 4. *Computation of the image of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\}) \cong \mathbb{Z}$ in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\}) \cong \mathbb{Z}^k$ for $n \geq 2$.*

By the splitting result mentioned earlier, it suffices to consider the maps $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n - \{p_j\})$ for each j, and the preceding discussion shows that these maps are isomorphisms which preserve canonical generators. Therefore the image of $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\}) \cong \mathbb{Z}$ in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\}) \cong \mathbb{Z}^k$ is merely the sum of the canonical free generators or the codomain.■

STEP 5. *Computation of the image of* $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$ *in* $H_{n-1}(F) \cong \mathbb{Z}^k$ *for* $n \geq 2$.

For each j such that $1 \leq j \leq k$, let $F_j = D^n - U_j$, where U_j is the small open disk centered at p_j. We shall begin by analyzing a commutative diagram which is related to the previous one:

$$\begin{array}{ccc}
H_n(\mathbb{R}^n, \mathbb{R}^n - \{p_1, \cdots, p_k\}) & \stackrel{\cong}{\longrightarrow} & \oplus_j \; H_n(\mathbb{R}^n, \mathbb{R}^n - \{p_j\}) \\
\downarrow \cong & & \downarrow \cong \\
H_{n-1}(\mathbb{R}^n - \{p_1, \cdots, p_k\}) & \longrightarrow & \oplus_j \; H_{n-1}(\mathbb{R}^n - \{p_j\}) \\
\uparrow \cong & & \uparrow \cong \\
H_{n-1}(F) & \longrightarrow & \oplus_j \; H_{n-1}(F_j)
\end{array}$$

The arrow in the first row is an isomorphism by excision, the arrows from the first to second rows are isomorphisms by the long exact homology sequences for the pairs (the adjacent terms in each case are positive dimensional homology groups of $\mathbb{R}^n$), and the arrows from the third to second rows are isomorphisms by Step 1. By the direct sum decompositions on the right, it suffices to analyze the image in homology when we are only removing the point p_j or the open disk U_j centered at p_j, where $1 \leq j \leq k$.

At this point we need to be careful about choosing the right signs for our free generators of homology groups, especially in view of the application we have in mind. If Σ is a sphere of radius r centered at $p \in \mathbb{R}^n$, we take the homeomorphism $S^{n-1} \to \Sigma$ constructed in Step 2: First stretch or shrink the sphere S^{n-1} of radius 1 centered at $\mathbf{0}$ to a concentric sphere of radius r, and then map this to the corresponding sphere centered at p via the translation $v \to v+p$. Then by the comments in the preceding paragraph we shall have proved the proposition if we can show the following:

> *Let $S_1 \subset \mathbb{R}^n$ be the sphere of radius a centered at p, suppose that the disk it bounds is contained in the interior of the disk of radius b centered at some point q, and let S_2 denote the boundary sphere of that disk. Let $f_1 : S^{n-1} \to S_1$ and $f : S^{n-1} \to S_2$ be the homeomorphisms given as above. Choose a generator ω of $H_{n-1}(S^{n-1})$. Then the images of $f_{1*}(\omega)$ and $f_{2*}(\omega)$ in $\mathbb{R}^n - \{p\}$ are equal.*

To prove this, let S_3 be the sphere of radius b centered at p, and let j_1 and j_3 denote the inclusions into $\mathbb{R}^n = \{p\}$. Then $j_3 \circ f_3 \simeq j_1 \circ f_1$ by the radial stretching homotopy sending (x,t) to $(1-t)\,f_1(x) \,+\, t\,f_3(x)$. Therefore $f_{1*}(\omega) = f_{3*}(\omega)$. By definition we also have $f_2(x) = f_3(x) + q - p$; if j_2 is th inclusion of S_2 in $\mathbb{R}^n - \{p\}$, it will suffice to prove that $j_3 \circ f_2 \simeq j_2 \circ f_2$, and this will follow if the image of the straight line homotopy $H(x,t) = f_3(x) + (1-t)(q-p)$ is contained in $\mathbb{R}^n - \{p\}$.

Since $|q-p| = d$ and a is the radius of the disk bounded by S_1, the condition that one sphere is contained in the interior of the open disk bounded by the other means that $d + a < b$ (Proof: If w is chosen so that $w - p$ is a negative multiple of $q - p$ and $|w - p| = a$, then $w \in S_1$, so that w is also in the open disk bounded by S_2 and therefore $b > |w-q| = |w-p| + |q-p| = a + d$). We need to show that $|x| = b$ implies that $H(x,t) \neq p$, or equivalently that $|x| = b$ implies that $|H(x,t) - p| > 0$. But we have

$$H(x,t) \; - \; p \;\; = \;\; f_3(x) \; - \; p \; + \; (1-t)\,(q-p)$$

which means that if $0 \leq t \leq 1$ (hence also $0 \leq 1-t \leq 1$) then

$$|H(x,t) - p| \quad \geq \quad |f_3(x) - p| \; - \; (1-t)\,|q-p| \quad > \quad b \;\; (1-t)\,d \quad \geq$$

$$b \; - \; d \quad > \quad a \quad > \quad 0$$

which is what we wanted to prove.■

A degree formula

We shall use the theorem to prove an abstract, multidimensional version of a result which plays a key role in complex analysis when $n = 2$.

COROLLARY. *Let F be as in the theorem, and suppose that we are given a continuous mapping $g : F \to \mathbb{R}^n - \{\mathbf{0}\}$, and for each j let E_j be the subdisk in D^n whose interior is removed to form F. Let $h_j : S^{n-1} \to F$ be the composite of the standard homeomorphism $S^{n-1} \to \mathrm{Bdy}\, E_j$ with the inclusion of $\mathrm{Bdy}\, E_j$ in F. Then we have a summation formula*

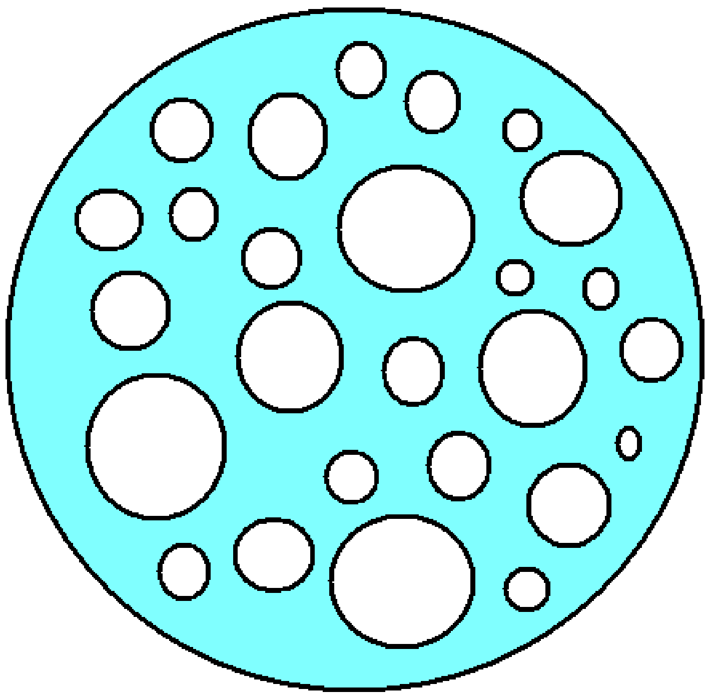
$$\deg\left(g|S^{n-1}\right) \quad = \quad \sum_j \, \deg\left(g \circ h_j\right) \; .$$

Section 4.5 of Ahlfors, *Complex Analysis* (Third Edition), describes implicatons of this result for complex function theory.

Proof of the corollary. Let ω be the standard generator of $H_{n-1}(S^{n-1})$ described in Step 5, and let $h_0 : S^{n-1} \to F$ be the inclusion mapping. Then by the theorem we have $h_{0*}(\omega) \; = \; \sum_j \; h_{j*}(\omega)$ and if we apply g_* to both sides we obtain a similar identity with h_{j_*} replaced by $g_* \circ h_{j_*} = (g \circ h_j)_*$ for all j. By the definition of degree we know that the image of ω under the latter map is equal to the degree of $g \circ h_j$ times ω if $j > 0$, and if $h = 0$ then the image of ω is equal to the degree of $g|S^{n-1}$ times ω.■

Drawing of a disk with holes

The following is a picture of a typical set F satisfying the conditions in the main theorem.



Note that the holes may be irregularly distributed throughout the disk and that the radii of the holes may also differ. Also, in general the center of the circle might not be one of the deleted center points. In the $2-$ dimensional case, the main theorem implies that outer circle with a counterclockwise parametrization is homologous to the sum of the inner circles with counterclockwise parametrizations (in other words, the two $1-$ dimensional configurations determine the same homology class).