II. Construction and uniqueness of singular homology

This unit proves the existence of a homology theory which satisfies nearly all the conditions formulated in Unit VI of `algtopnotes2012.tex`. The following summarizing table provides more precise references:

Axiom Type	Axiom Numbers	Pages
Primitive Data	(T.1)–(T.5)	74–75
Functoriality and naturality	(A.1)–(A.6)	75–77
Exactness	(B.1)–(B.3)	77–78
Homotopy Invariance	(C.1)	79
Compact/Polyhedral Generation	(C.2)–(C.3)	79–80
Normalization	(D.1)–(D.5)	80–81
Excision	(E.1)–(E.2)	82
Mayer-Vietoris Sequences	(E.3)–(E.4)	82–83

The basic idea of the existence proof is very simple: We modify the construction of simplicial chain complexes to obtain a new functor from the category of topological spaces to the category of chain complexes, and we take the homology groups of these chain complexes. By functoriality, such groups will automatically be topologically invariant. Many steps in verifying the axioms will be fairly straightforward, but there are two crucial pieces of input from Unit I of these notes that will be needed:

(1) In Section I.5 we constructed a chain $P_{q+1} \in C_{q+1}(\Delta_q \times [0,1])$ which was an integral linear combination of all the simplices in $\Delta_q \times [0,1]$ with coefficients ± 1. This chain will be used to show that homotopic maps of spaces define chain homotopic maps of chain complexes, which will imply that the homotopic maps induce the same mappings in homology.

(2) Given an open covering $\mathcal{U}$ of a space X, it is sometimes necessary to know that we can somehow replace an algebraic chain for X by another chain whose pieces are so small that each one lies inside a set in the open covering. If we are dealing with simplicial chains over a simplicial complex, this can be done using iterated barycentric subdivisions. Historically speaking, one of the most important steps in the development of singular homology theory was to "leverage" barycentric subdivision into a construction for singular homology.

In the final section of this unit we shall prove uniqueness theorems for constructions satisfying all the axioms for singular homology described in Unit VI of `algtopnotes2012.tex` except for (D.5), which relates the fundamental group of an arcwise connected space to its 1-dimensional homology; the statement of this axiom assumes the existence of certain natural transformations relating fundamental groups and homology, and the uniqueness results do not require any of this structure. In Unit III we shall construct these natural transformations from the fundamental group functor to the singular homology theory constructed here, and we shall verify the axiom relating the fundamental group to 1-dimensional homology.

It took about a half century for mathematicians to come up with the formulation that is now standard, starting with Poincaré's initial papers on topology (which he called *analysis situs*) at the end of the 19^{th} century and culminating with the definition of *singular homology* by S. Eilenberg and N. Steenrod in the nineteen forties (with many important contributions by others along the way).

Some books start directly with singular homology and do not bother to develop simplicial homology. The reason for considering the latter here is that it is in some sense a “toy model” of singular homology for which many basic ideas appear in a more simplified framework.

II.1 : Basic definitions and properties

(Hatcher, §§ 2.1, 2.3)

As before, let Δ_q be the standard q-simplex in $\mathbb{R}^{q+1}$ whose vertices are the standard unit vectors $\mathbf{e}_0, \cdots, \mathbf{e}_q$. If $(P, \mathbf{K})$ is a simplicial complex, then for each free generator $\mathbf{v}_0 \cdots \mathbf{v}_q$ of $C_q(P, \mathbf{K})$ there is a unique affine (hence continuous) map $T : \Delta_q \to P$ which sends a point $(t_0, \cdots, t_q) \in \Delta_{q+1}$ to $\sum_j t_j \mathbf{v}_j \in P$. One can think of these as linear simplices in P. The idea of singular homology is to consider more general continuous mappings from Δ_q to a space X, viewing them as simplices with possible singularities or *singular simplices* in the space.

Definition. Let X be a topological space. A *singular q-simplex* in X is a continuous mapping $T : \Delta_q \to X$, and the abelian group of *singular q-chains* $S_q(X)$ is defined to be the free abelian group on the set of singular q-simplices.

If we let $\partial_j : \Delta_{q-1} \to \Delta_q$ be the affine map which sends Δ_{q-1} to the face opposite the vertex $\mathbf{e}_j$ and is order preserving on the vertices, then as in the case of simplicial chains we have boundary homomorphisms $d_q : S_q(X) \to S_{q-1}(X)$ given on generators by the standard formula:

$$d_q(T) \quad = \quad \sum_{j=0}^{n} (-1)^i \partial_i(T) \quad = \quad \sum_{j=0}^{n} (-1)^i T \circ \partial_i$$

Likewise, there are augmentation maps $\varepsilon : S_0(X) \to \mathbb{Z}$ which send each free generator $T : \Delta_0 \to X$ to $1 \in \mathbb{Z}$.

We then have the following results:

PROPOSITION 1. *The homomorphisms d_q make $S_*(X)$ into a chain complex, and if $(P, \mathbf{K})$ is a simplicial complex, then the affine map construction makes $C_*(P, \mathbf{K})$ into a chain subcomplex of $S_q(P)$, and the inclusion is augmentation preserving. Furthermore, if A is a subset of X, then $S_*(A)$ is canonically identified with a subcomplex of $S_*(X)$ by the map taking $T : \Delta_q \to X$ into $i \circ T : \Delta_q \to X$, where $i : A \to X$ is the inclusion mapping.*■

PROPOSITION 2. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Then there is a chain map $f_\#$ from $S_*(X)$ to $S_*(Y)$ such that for each singular q-simplex T the value $f_\#(T)$ is given by $f \circ T$. This construction transforms the singular chain complex construction into a covariant functor from topological spaces and continuous maps to chain complexes (and chain maps). Furthermore, passage to quotients yields a covariant functor from pairs of topological spaces and continuous maps of pairs to chain complexes and chain maps.*

This is essentially an elementary verification, and probably the most noteworthy part is the need to verify that $f_\#$ is a chain map. Details are left to the reader.■

Predictably, the homology groups we want are the homology groups of the singular chain complexes.

Definition. If X is a topological space, then the *singular homology groups* $H_*(X)$ are the corresponding homology groups of the chain complex defined by $S_*(X)$. More generally, if A is a subset of X, then the *relative chain complex* $S_*(X, A)$ is defined to be $S_*(X)/S_*(A)$, and the *relative singular homology groups* $H_*(X, A)$ are the corresponding homology groups of that quotient complex. Note that if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then Proposition 1 generalizes to yield a chain map from $\theta_\#$: $C_*(\mathbf{K}, \mathbf{L})$ to $S_*(P, Q)$. — Note that the relative groups (both singular and simplicial) do not have augmentation homomorphisms if A or $\mathbf{L}$ is nonempty.

It is not difficult to show that the singular homology groups of homeomorphic spaces are isomorphic, and in fact it is an immediate consequence of the following results:

PROPOSITION 3. *The homology groups $H_*(X, A)$ and homomorphisms $f_*; H_*(X, A) \to H_*(Y, B)$ define a covariant functor from the category of pairs of topological spaces to the category of abelian groups and homomorphisms. Furthermore, if $(\mathbf{K}, \mathbf{L})$ is a pair consisting of a simplicial complex and a subcomplex with underlying space pair (P, Q), then the chain map $\theta_\#$ induces a natural transformation of functors $\theta_* : H_*(\mathbf{K}, \mathbf{L}) \to H_*(P, Q)$.*■

This proposition shows that we have data types (T.3) and (T.5) in our axiomatic description of singular homology, and it also verifies axioms (A.1) and (A.2), which involve functoriality and naturality with respect to simplicial homology.

Since functors send isomorphisms in source category to isomorphisms in the target, the topological invariance of singular homology groups is a trivial consequence of Proposition 3.

COROLLARY 4. *If X and Y are topological spaces and $f : X \to Y$ is a homeomorphism, then the associated homomorphism of graded homology groups $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*■

By Corollary 3, the simplicial homology groups of homeomorphic polyhedra will be isomorphic if we can give an affirmative answer to the following question for all simplicial complexes $(P, \mathbf{K})$:

PROBLEM. If $(P, \mathbf{K})$ is a simplicial complex and $\lambda : C_*(\mathbf{K}) \to S_*(P)$ is the associated chain map, does $\theta_* : H_*(\mathbf{K}) \to H_*(P)$ define an isomorphism of homology groups?

We shall prove this later. For the time being we note that *the map λ is a chain level isomorphism if $\mathbf{K}$ is given by a single vertex* (in this case each of the groups $S_q(X)$ is cyclic, and it is generated by the constant map from Δ_q to X).

The simplest normalization properties of homology groups

It will be convenient to go through the verifications roughly in order of increasing complexity rather than to follow the ordering given in `algtopnotes2012.pdf`. From this viewpoint, the next axioms to consider are the normalization axioms (D.2)–(D.4); it is mildly ironic that (D.1) will be one of the last axioms to be verified.

The verification of (D.4), which states that negative-dimensional homology groups are zero, is particularly tirival; the simplicial chain groups $S_q(X, A)$ vanish by construction if $q < 0$, and since the homology groups are subquotients of the chain groups they must also vanish.

If X is a topological space and $T : \Delta_q \to X$ is a singular simplex, then the image of T lies entirely in a single path component of X. Therefore the next result, whose conclusion includes the statement of (D.2), follows immediately.

PROPOSITION 5. *If X is a topological space and its path components are the subspaces X_α, then the maps $S_*(X_\alpha)$ to $S_*(X)$ induced by inclusion define an isomorphism of chain complexes $\bigoplus S_*(X_\alpha) \to S_*(X)$ and hence also a homology isomorphism from $\bigoplus H_*(X_\alpha)$ to $H_*(X)$.*■

The preceding results lead directly to a verification of (D.3).

COROLLARY 6. *In the setting above, $H_0(X)$ is isomorphic to the free abelian group on the set of path components of X.*

A proof of this result is given on pages 109 – 110 of Hatcher.■

One immediate consequence of the preceding observations is that the map from $C_*(\mathbf{K})$ to $S_*(P)$ is an isomorphism if $(P, \mathbf{K})$ is 0-dimensional, and similarly for the map from $H_*(\mathbf{K})$ to $H_*(P)$.

Although we are far from ready to verify (D.1) in complete generality, we can do so for the very simplest examples.

PROPOSITION 7. (The Eilenberg-Steenrod Dimension Axiom) *If $X = \{x\}$ consists of a single point, then $H_q(X) = 0$ if $q \neq 0$, and $H_0(X) \cong \mathbb{Z}$ with the isomorphism given by the augmentation map.*

Proof. Suppose first that $x \in \mathbb{R}^n$ for some n, so that $\{x\}$ is naturally a 0-dimensional polyhedron. We have already noted that the simplicial and singular chains on X are isomorphic. Since the conclusion of the proposition holds for (unordered) simplicial chains by the results of the preceding unit, it follows that the same holds for singular chains. To prove the general case, note that if $\{x\}$ is an arbitrary space consisting of a single point and $\mathbf{0} \in \mathbb{R}^n$, then $\{\mathbf{0}\}$ is homeomorphic to $\{x\}$ and in this case the conclusion follows from the special case because homeomorphic spaces have isomorphic homology groups.■

The compact supports property

Our next result verifies (C.2) and is often summarized with the phrase, *singular homology is compactly supported*. This was not one of the original Eilenberg-Steenrod axioms, but its importance for using singular homology was already clear when Eilenberg and Steenrod developed singular homology.

THEOREM 8. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subspace $A \subset X$ such that u lies in the image of the associated map from $H_q(A)$ to $H_q(X)$. Furthermore, if A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$, then there is a compact subset B satisfying $A \subset B \subset X$ such that the images of u and v are equal in $H_q(B)$.*

Proof. If c is a singular q-chain and

$$c \;=\; \sum_j n_j\, T_j$$

define the *support* of c, written $\mathsf{Supp}\,(c)$, to be the compact set $\cup_j\, T_j(\Delta_q)$. Note that this subset is compact.

If $u \in H_q(X)$ is represented by the chain z and if $A = \mathsf{Supp}\,(z)$, then since $S_*(A) \to S_*(X)$ is 1–1 it follows that z represents a cycle in A and hence u lies in the image of $H_q(A) \to H_q(X)$.

Suppose now that A is a compact subset of X and $u, v \in H_q(A)$ are two classes with the same image in $H_q(X)$. Let z and w be chains in $S_q(A)$ representing u and v respectively, and let $b \in S_{q+1}(X)$ be such that $d(b) = i_\#(z) - i_\#(w)$. If we set $B = A \cup \mathsf{Supp}\,(b)$, then it follows that the images of $z - w$ bounds in $S_q(B)$, and therefore it follows that u and v have the same image in $H_q(B)$.■

II.2 : Exactness and homotopy invariance

(Hatcher, §§ 2.1, 2.3)

We have seen that long exact sequences and homotopy invariance yield a great deal of information about homology groups. The next step is to verify some of the properties for singular homology and their compatibility with the analogous properties for simplicial homology.

The exact sequence of a pair

In 205B the long exact sequence of a pair in simplicial homology turned out to be a direct consequence of the corresponding long exact homology sequence for a short exact sequence of chain complexes. In view of our definitions, it is not surprising that the same considerations yield long exact sequences of pairs in singular homology.

THEOREM 1. (Long Exact Homology Sequence Theorem — Singular Homology Version). *Let (X, A) be a pair of topological spaces where A is a subspace of X. Then there is a long exact sequence of homology groups as follows:*

$$\cdots \quad H_{k+1}(X,A) \xrightarrow{\partial} H_k(A) \xrightarrow{i_*} H_k(X) \xrightarrow{j_*} H_k(X,A) \xrightarrow{\partial} H_{k-1}(A) \quad \cdots$$

This sequence extends indefinitely to the left and right. Furthermore, if we are given another pair of spaces (Y, B) and a continuous map of pairs $f : (X, A) \to (Y, B)$ such that $f : X \to Y$ is continuous and $f[A] \subset B$, then we have the following commutative diagram in which the two rows are exact:

$$\begin{array}{ccccccccccc}
\cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots \\
\cdots & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & & \Big\downarrow f_* & \\
\cdots & H_{k+1}(Y,B) & \xrightarrow{\partial'} & H_k(B) & \xrightarrow{i'_*} & H_k(Y) & \xrightarrow{j'_*} & H_k(Y,B) & \xrightarrow{\partial'} & H_{k-1}(B) & \cdots
\end{array}$$

This follows immediately from the algebraic theorem on long exact homology sequences and the definitions of the various homology groups in terms of a short exact sequence of chain complexes.∎

There is also a map of long exact sequences relating simplicial and singular homology for simplicial complexes. This is not one of the Eilenberg-Steenrod properties, but logically it fits naturally into the discussion here.

THEOREM 2. *Let $(X, \mathbf{K})$ be a simplicial complex, and let $(A, \mathbf{L})$ determine a subcomplex. Then there is a commutative ladder as below in which the horizontal lines represent the long exact homology sequences of pairs and the vertical maps are the natural transformations from simplicial to singular homology.*

$$\begin{array}{ccccccccccc}
\cdots & H_{k+1}(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_k(\mathbf{L}) & \xrightarrow{i_*} & H_k(\mathbf{K}) & \xrightarrow{j_*} & H_k(\mathbf{K},\mathbf{L}) & \xrightarrow{\partial} & H_{k-1}(\mathbf{L}) & \cdots \\
\cdots & \Big\downarrow \lambda_* & & \Big\downarrow \lambda_* & & \Big\downarrow \lambda_* & & \Big\downarrow \lambda_* & & \Big\downarrow \lambda_* & \\
\cdots & H_{k+1}(X,A) & \xrightarrow{\partial} & H_k(A) & \xrightarrow{i_*} & H_k(X) & \xrightarrow{j_*} & H_k(X,A) & \xrightarrow{\partial} & H_{k-1}(A) & \cdots
\end{array}$$

The results follow directly from the Five Lemma and the fact that the previously defined chain maps λ pass to morphisms of quotient complexes of relative chains from $C_*(\mathbf{K},\mathbf{L})$ to $S_*(X, A)$.∎

Theorems 1 and 2 combine to show that our construction has several of the necessary properties for an abstract singular homology theory; namely, it yields data types (T.2) and (T.5) and axioms (A.2)–(A.3), (A.5) and (B-1)–(B.3). The remainder of this section is devoted to verifying axiom (C.1), and thus the results of this section reduce the verification of singular homology axioms to the following:

(1) Construction of data type (T.2).

(2) Verification of axioms (A.4), (D.1) and (E.1)–(E.4).

(3) Construction of data type (T.4), and verification of axioms (A.6), (C.3) and (D.5).

We shall take care of the first two points in Sections II.3 and II.4. This will prove that one has a theory with all the properties needed to derive the applications in Unit VII in `algtop-notes2012.pdf`. Axiom (C.3) will be needed to prove the uniqueness results for axiomatic singular homology in Section II.5, and a reader who wishes to skip this may do so without loss of continuity. Finally, data type (T.4), and axioms (A.6) and (D.5) are not needed to prove uniqueness, and we are postponing the discussion of these features until the next unit.

Homotopy invariance

By definition, two maps of topological space pairs $f, g : (X, A) \to (Y, B)$ are homotopic *as maps of pairs* if there is a homotopy $H : (X \times [0,1], A \times [0,1]) \to (Y, B)$ such that the restriction of H to $(X \times \{0\}, A \times \{0\})$ and $(X \times \{1\}, A \times \{1\})$ are given by f and g respectively

The discussion of chain homotopies in Section I.5 suggests the following question: *If f and g are homotopic maps from (X, A) to (Y, B), will the associated chain maps from $S_q(X, A)$ to $S_q(Y, B)$ be chain homotopic?*

An affirmative answer to this question implies axiom (C.1), which states that homotopic maps of pairs induce the same homomorphisms in singular homology. The next result confirms that the answer to the preceding question is yes.

THEOREM 3. (Homotopy invariance of singular homology) *Suppose that $f, g : (X, A) \to (Y, B)$ are homotopic as maps of pairs. Then the associated chain maps $f_\#, g_\# : S_*(X, A) \to s_*(Y, B)$ are chain homotopic, and the associated homology homomorphisms $f_*, g_* : H_*(X, A) \to H_*(Y, B)$ are equal.*

Before proving this result, we shall state three important consequences.

COROLLARY 4. *If $f : X \to Y$ is a homotopy equivalence, then the associated homology maps $f_* : H_*(X) \to H_*(Y)$ are isomorphisms.*

Proof. Let $g : Y \to X$ be a homotopy inverse to f. Since $g \circ f$ is homotopic to the identity on X and $g \circ g$ is homotopic to the identity on Y, it follows that the composites of the homology maps $g_* \circ f_*$ and $f_* \circ g_*$ are equal to the identity maps on $H_*(X)$ and $H_*(Y)$ respectively, and therefore f_* and g_* are isomorphisms.■

COROLLARY 5. *If X is a contractible space and there is a contracting homotopy from the identity to the constant map whose value is given by $y \in X$, then the inclusion of $\{y\}$ in X defines an isomorphism of singular homology groups.*

Proof. Let $i : \{y\} \to X$ be the inclusion map, and let $r : X \to \{y\}$ be the constant map, so that $r \circ i$ is the identity. The contracting homotopy is in fact a homotopy from the identity to the reverse composite $i \circ r$, and therefore $\{y\}$ is a deformation retract of X. By the preceding corollary, it follows that i_* defines an isomorphism of singular homology groups.■

COROLLARY 6. *If $f : (X, A) \to (Y, B)$ is a continuous map of pairs such that the associated maps $X \to Y$ and $A \to B$ are homotopy equivalences, then the homology maps f_* from $H_*(X, A)$ to $H_*(Y, B)$ all isomorphisms.*

Proof. In this case we have a commutative ladder as in Theorem 1, in which the horizontal lines represent the exact homology sequences of (X, A) and (Y, B), while the vertical arrows represent the homology maps defined by the mapping f. Since the mappings from X to Y and from A to B are homotopy equivalences, it follows that all the vertical maps except possibly those involving $H_*(X, A) \to H_*(Y, B)$ are isomorphisms; one can now use the Five Lemma to prove that these remaining vertical maps are also isomorphisms.■

The following simple observation will be useful in the proof of Theorem 3:

LEMMA 7. *For each $t \in [0, 1]$ let $i_t : X \to X \times [0, 1]$ denote the slice inclusion $i_t(x) = (x, t)$, Then i_0 and i_1 are homotopic.*

Proof. The identity map on $X \times [0, 1]$ defines a homotopy from i_0 to i_1.■

Proof of Theorem 3. We shall first show that it suffices to prove the theorem for the mappings i_0 and i_1 described in Lemma 7. For suppose we have continuous mappings $f, g : X \to Y$ and a homotopy $H : X \times [0, 1] \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$. Then we also have

$$f_* \;=\; (H \circ i_0)_* \;=\; H_* \circ (i_0)_* \;=\; H_* \circ (i_1)_* \;=\; (H \circ i_1)_* \;=\; g_*$$

showing that f and g define the same maps in homology.

To prove the result for the mappings in Lemma 7 we shall in fact prove that *the chain maps $(i_0)_\#$ and $(i_1)_\#$ from $S_*(X)$ to $S_*(X \times [0, 1])$ are chain homotopic.* — The results of Section I.5 will then imply that the homology maps $(i_0)_*$ and $(i_1)_*$ are equal.

In Section I.5 we noted the existence of simplicial chains $P_{q+1} \in C_{q+1}(\Delta_q \times [0, 1])$ such that $P_0 = 0$, $P_1 = \mathbf{y}_0\mathbf{x}_0$ and more generally

$$dP_{q+1} \;=\; (i_1)_\#\mathbf{1}_q \;-\; (i_0)_\#\mathbf{1}_q \;-\; \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

where $\mathbf{1}_q = \mathbf{e_0} \;\cdots\; \mathbf{e}_q \in C_q(\Delta_q)$, the map $\partial_j = \partial_j^{[q]} : \Delta_{q-1} \to \Delta_q$ is affine linear onto the face opposite $\mathbf{e}_j$, and $(-)_\#$ generically denotes an associated chain map. Recall that the existence of the chains P_{q+1} was proved inductively, the key point being that since $\Delta_q \times \mathbf{I}$ is acyclic, such a chain exists if the boundary of

$$(i_1)_\#\mathbf{1}_q \;-\; (i_0)_\#\mathbf{1}_q \;-\; \sum_j (-1)^j (\partial_j \times 1)_\# P_q$$

is equal to zero.

To construct the chain homotopy $K : S_q(X) \to S_{q+1}(X \times [0, 1]$, let $T : \Delta_q \to X$ be a free generator of $S_q(X)$ and set $K(T) = (T \times \mathrm{id}_{[0,1]})_\# P_{q+1}$. We then have

$$d\,K(T) \;=\; d \circ (T \times \mathrm{id}_{[0,1]})_\# P_{q+1} \;=\; (T \times \mathrm{id}_{[0,1]})_\# \circ d(P_{q+1}) \;=\;$$

$$(T \times 1)_{\#} \circ (i_1)_{\#} \mathbf{1}_q \ - \ (T \times 1)_{\#} \circ (i_0)_{\#} \mathbf{1}_q \ - \ \sum_j \ (-1)^j d \circ (T \circ \partial_j \times 1)_{\#} P_q \ =$$

$$(i_1)_{\#} \circ T_{\#}(\mathbf{1}_q) \ - \ (i_0)_{\#} \circ T_{\#}(\mathbf{1}_q) \ - \ \sum_j \ (-1)^j (T \circ \partial_j \times 1)_{\#} d(P_q) \ =$$

$$(i_1)_{\#}(T) \ - \ (i_0)_{\#}(T) \ - \ K \circ d(T) \ .$$

Therefore K defines a chain homotopy between $(i_1)_{\#}$ and $(i_0)_{\#}$ as required.∎

II.3 : Excision and Mayer-Vietoris sequences

(Hatcher, §§ 2.1 – 2.3)

The final Eilenberg-Steenrod axiom, called *excision*, is the most complicated to state and to prove, and the crucial steps in the argument trace back to the proofs of the following two results in simplicial homology theory:

(1) If the polyhedron P is obtained from the polyhedron Q by adjoining a single simplex S (whose boundary must lie in Q), then the inclusion from $(S, \partial S)$ to (P, Q) defines an isomorphism in simplicial homology. More generally, if P_1 and P_2 correspond to subcomplexes of P in some simplicial decomposition and $P = P_1 \cup P_2$, then the inclusion map from $(P_1, P_1 \cap P_2)$ to $(P = P_1 \cup P_2, P_2)$ defines isomorphisms in homology.

(2) For every simplicial complex $(P, \mathbf{K})$, the homology groups of $(P, \mathbf{K})$ and its barycentric subdivision $\big(P, B(\mathbf{K})\big)$ are naturally isomorphic (with respect to subcomplex inclusions).

In particular, the excision axioms are essentially abstract, highly generalized versions of statement (1), both in terms of their formulations and their proofs. Usually the following restatement of (E.2) is taken to be the main version of excision.

THEOREM 1. *Suppose that (X, A) is a topological space and that U is a subset of X such that $U \subset \overline{U} \subset$ Interior(A). Then the inclusion map from $(X - U, A - U)$ to (X, A) determines isomorphisms in homology.*

Here is the analogous restatement of (E.1).

THEOREM 2. *Suppose that the space X can be written as a union of subsets $A \cup B$ such that the interiors of A and B form an open covering of X. Then the inclusion of pairs from $(B, A \cap B)$ to $(X = A \cup B, A)$ induces isomorphisms in homology.*

In particular, *the conclusion of Theorem* 2 *is valid if both A and B are open subsets of X.*

One can derive Theorem 1 as a consequence of Theorem 2 by taking $B = X - U$ (note that the open set $X - \overline{U}$ is contained in $X - U$).∎

There is an obvious formal similarity involving the most general statement in (1), the statement of (E.1) in Theorem 2, and the standard module isomorphism

$$M/M \cap N \quad \cong \quad M + N/N\ x \quad \text{(where} \quad M \text{ and } N \quad \text{are submodules of some module} \quad L)$$

and we shall see that the similarities are more than just a coincidence.

Using the acyclicity of $C_*(\Delta_q)$ we may inductively construct chains $\beta_q \in C_q\big(B(\Delta_q)\big)$ (simplicial chains on the barycentric subdivision) such that $\beta_0 = \mathbf{1}_0$ and

$$d(\beta_q) \quad = \quad \sum_j \,(-1)^j\,(\partial_j)_\#\beta_{q-1}$$

for $q \geq 0$. If X is a topological space, then we may define a graded homomorphism $\beta : S_*(X) \to S_*(X)$ such that for each singular simplex $T : \Delta_q \to X$ we have $\beta(T) = T_\#(\beta_q)$.

LEMMA 3. *The graded homomorphism β is a map of chain complexes. Furthermore, if A is a subspace of X then β maps $S_*(A)$ into itself.*

Proof. We have $d\circ\beta(T) = d\circ T_\#(\beta_q) = T_\#\circ d(\beta_q)$, and the last term is equal to

$$T_\# \left(\sum_j \,(-1)^j\,(\partial_j)_\#\beta_{q-1} \right) \quad = \quad \sum_j \,(-1)^j\,(T\circ\partial_j)_\#\beta_{q-1}$$

which in turn is equal to $\beta\big(d(T)\big)$.■

The significance of the barycentric subdivision chain map is that it takes a chain in a given homology class and replaces it by a chain which is in the same homology class but is composed of smaller pieces; in fact, if one applies barycentric subdivision sufficiently many times, it is possible to find a chain representing the same homology class such that its chain are arbitrarily small. Justifications of these assertions will require several steps.

The first objective is to prove that the barycentric subdivision map is chain homotopic to the identity. As in previous constructions, this begins with the description of some universal examples.

PROPOSITION 4. *There are singular chains $L_{q+1} \in S_{q+1}(\Delta_n)$ such that $L_1 = 0$ and $d(L_{q+1}) = \beta_q - \mathbf{1}_q - \sum_j \,(-1)^j(\partial_j)_\#(L_q)$.*

By convention we take $L_0 = 0$.

Sketch of proof. Once again, the idea is to construct the chains recursively. Since Δ_q is acyclic, we can find a chain with the desired properties provided the difference

$$\beta_q \;-\; \mathbf{1}_q \;-\; \sum_j \,(-1)^j(\partial_j)_\#(L_q)$$

is a cycle. One can prove this chain lies in the kernel of d_q by using the recursive formulas for $d_q(\beta_q)$, $d_q(\mathbf{1}_q)$, and $d_q(L_q)$.■

PROPOSITION 5. *If X is a topological space and $A \subset X$ is a subspace, then the identity and the barycentric subdivision maps on $S_*(X, A)$ are chain homotopic.*

Proof. It will suffice to construct a chain homotopy on $S_*(X)$ that sends the subcomplex $S_*(A)$ to itself, for one can then obtain the relative statement by passage to quotients.

Define homomorphisms $W : S_q(X) \to S_{q+1}(X)$ on the standard free generators $T : \Delta_q \to X$ by the formula

$$W(T) \quad = \quad T_\# L_{q+1} \; .$$

By construction, if $T \in S_q(A)$ then $W(T) \in S_{q+1}(A)$. The proof that W is a chain homotopy uses the recursive formula for L_{q+1} and is entirely analogous to the proof that the map K in the proof of Theorem ????? is a chain homotopy.■

Small singular chains

We have noted that barycentric subdivision takes a cycle and replaces it by a homologous cycle composed of smaller pieces and that if one iterates this procedure then one obtains a chain whose pieces are arbitrarily small. Not surprisingly, we need to formulate this more precisely.

Definition. Let X be a topological space, and let $\mathcal{F}$ be a family of subsets whose interiors form an open covering of X. A singular chain $\sum_i n_i T_i \in S_q(X)$ is said to be $\mathcal{F}$*-small* if for each i the image $T_i(\Delta_q)$ lies in a member of $\mathcal{F}$. Denote the subgroup of $\mathcal{F}$-small singular chains by $S_*^{\mathcal{F}}(X)$. It follows immediately that the latter is a chain subcomplex of $S_*^{\mathcal{F}}(X)$; furthermore, if $A \subset X$ and we define $S_*^{\mathcal{F}}(A)$ to be the intersection of $S_*^{\mathcal{F}}(X)$ and $S_*^{\mathcal{F}}(A)$, then we may define relative $\mathcal{F}$-small chain groups of the form

$$S_*^{\mathcal{F}}(X,A) \quad = \quad S_*^{\mathcal{F}}(X)/S_*^{\mathcal{F}}(A) \ .$$

Note further that the barycentric subdivision maps send $\mathcal{F}$-small chains into $\mathcal{F}$-small chains.

THEOREM 6. *For all (X,A) and $\mathcal{F}$, the inclusion mappings $S_*^{\mathcal{F}}(X,A) \to S_*(X,A)$ define isomorphisms in homology.*

Proof. It is a straightforward algebraic exercise to prove that if L is a chain homotopy from the barycentric subdivision map β to the identity, then for each $r \geq 1$ the map $(1 + \ \cdots \ + \beta^{r-1}) \circ L$ defines a chain homotopy from β^r to the identity.

Let $\mathcal{U}$ be the open covering of X obtained by taking the interiors of the sets in $\mathcal{F}$.

Suppose first that we have $u \in H_*(X,A)$ and u is represented by the cycle $z \in S_*(X,A)$. Write $z = \sum_i n_i T_i$ and construct open coverings $\mathcal{W}_i$ of Δ_q by $\mathcal{W}_i = T_i^{-1}(\Delta_q)$. Then by the Lebesgue Covering Lemma there is a positive integer r such that for each i, every simplex in the r^{th} barycentric subdivision of Δ_q lies in a member of $\mathcal{W}_i$. It follows immediately that $\beta^r(z)$ is $\mathcal{F}$-small. Since β^r is a chain map, it follows that $\beta^r(z)$ is also a cycle in both $S_*(X,A)$ and the subcomplex $S_*^{\mathcal{F}}(X,A)$, and since β is chain homotopic to the identity it follows that

$$u \quad = \quad \beta_*(u) \quad = \quad \cdots \quad = \quad (\beta_*)^r(u) \quad = \quad (\beta^r)_*(u)$$

and hence u lies in the image of the homology of the small singular chain group.

To complete the proof we must show that if two cycles in $S_*^{\mathcal{F}}(X,A)$ are homologous in $S_*(X,A)$ then they are also homologous in $S_*^{\mathcal{F}}(X,A)$. Let z_1 and z_2 be the cycles, and let $dw = z_2 - z_1$ in $S_*(X,A)$. As in the preceding paragraph there is some t such that $\beta^t(w) \in S_*^{\mathcal{F}}(X,A)$. Since β^t is a chain map and is chain homotopic to the identity, it follows that we have

$$[z_2] \quad = \quad (\beta^t)_*[z_2] \quad = \quad [\beta^t(z_2)] \quad = \quad [\beta^t(z_1)] \quad = \quad (\beta^t)_*[z_1] \quad = \quad [z_1]$$

in the $\mathcal{F}$-small homology $H_*^{\mathcal{F}}(X,A)$. Therefore we have shown that the map from $H_*^{\mathcal{F}}(X,A)$ to $H_*(X,A)$ is also injective, and hence it must be an isomorphism.■

Application to Excision

We recall the hypotheses of the Excision Property: A pair of topological spaces (X, A) is given, and we have an open subset $U \subset X$ such that $\overline{U} \subset \mathrm{Int}(A)$. Excision then states that the inclusion map of pairs from $(X - U, A - U)$ to (X, A) defines isomorphisms of singular homology groups.

Predictably, we shall use the previous results on small chains. Let $\mathcal{F}$ be the family of subsets given by A and $X - U$. Then by the hypotheses we know that the interiors of the sets in $\mathcal{F}$ form an open covering of X, and by definition the subcomplex $S_*^{\mathcal{F}}(X)$ is equal to $S_*(A) + S_*(X - U)$. Therefore the chain level inclusion map from $S_*(X - U, A - U)$ to $S_*(X, A)$ may be factored as follows:

$$S_*(X-U, A-U) \quad = \quad S_*(X-U)/S_*(A-U) \quad = \quad S_*(X-U)/\left(\, S_*(A) \cap S_*(X-U)\,\right) \quad \longrightarrow$$

$$\left(\, S_*(A) + S_*(X-U)\,\right)/S_*(A) \quad = \quad S_*^{\mathcal{F}}(X, A) \quad \subset \quad S_*(X, A)$$

Standard results in group theory imply that the last morphism on the top line is an isomorphism, and the preceding theorem shows that the last morphism on the second line is an isomorphism. Therefore if we pass to homology we obtain an isomorphism from $H_*(X - U, A - U)$ to $H_*(X, A)$, which is precisely the statement of the Excision Property.■

The same methods also yield the following result:

PROPOSITION 7. *If U and V are open subsets of a topological space, then the maps in singular homology induced by the inclusions $(U, U \cap V) \subset (U \cup V, V)$ are isomorphisms.*■

Axioms (E.1) and (E.2) follow immediately from the preceding discussion.

Mayer-Vietoris sequences

One of the most useful results for computing fundamental groups is the Seifert-van Kampen Theorem. There is a similar principle that can be applied to find the homology groups of a space X presented as the union of two open subsets U and V; in fact, the result in homology does not require any connectedness hypotheses on the intersection.

THEOREM 8. (Mayer-Vietoris Sequence for open sets in singular homology.) *Let X be a topological space, and let U and V be open subsets such that $X = U \cup V$. Denote the inclusions of U and V in X by i_U and i_v respectively, and denote the inclusions of $U \cap V$ in U and V by g_U and g_V respectively. Then there is a long exact sequence*

$$\cdots \to H_{q+1}(X) \to H_q(U \cap V) \to H_q(U) \oplus H_q(V) \to H_q(X) \to \cdots$$

in which the map from $H_*(U) \oplus H_*(V)$ to $H_*(X)$ is given on the summands by $(i_U)_*$ and $(i_V)_*$ respectively, and the map from $H_*(U \cap V)$ to $H_*(U) \oplus H_*(V)$ is given on the factors by $-(g_U)_*$ and $(g_V)_*$ respectively (note the signs!!).

Theorem 8 yields data type (T.2) and axiom (E.3) for singular homology.

Proof. Let $\mathcal{U}$ be the open covering of X whose sets are U and V, and let $S_*^{\mathcal{U}}(X)$ be the chain complex of all $\mathcal{U}$-small chains in $S_*(X)$. Then we have

$$S_*^{\mathcal{U}}(X) \quad = \quad S_*(U) \; + \; S_*(V) \quad \subset S_*(X)$$

(note that the sum is not direct) and hence we also have the following short exact sequence of chain complexes, in which the injection is given by the chain map whose coordinates are $-(g_U)_\#$ and $(g_V)_\#$ and the surjection is given on the respective summands by $(i_U)_\#$ and $(i_V)_\#$:

$$0 \longrightarrow S_*(U \cap V) \longrightarrow S_*(U) \oplus S_*(V) \longrightarrow S_*^{\mathcal{U}}(X) \longrightarrow 0$$

The Mayer-Vietoris sequence is the long exact homology sequence associated to this short exact sequence of chain complexes combined with the isomorphism $H_*^{\mathcal{U}}(X) \cong H_*(X)$.■

We have noted that one also has a Mayer-Vietoris sequences in simplicial homology, but for much different types of subspaces (in particular, the assumption is that a polyhedron is the union of two subcomplexes, and every subcomplex is closed and usually not open in P). Specifically, if $\mathbf{K}_1$ and $\mathbf{K}_2$ are subcomplexes of some $\mathbf{K}$, where $(P, \mathbf{K})$ is a simplicial complex, then the corresponding Mayer-Vietoris sequence has the following form:

$$\cdots \to H_{q+1}(\mathbf{K}) \to H_q(\mathbf{K}_1 \cap \mathbf{K}_2) \to H_q(\mathbf{K}_1) \oplus H_q(\mathbf{K}_2) \to H_q(\mathbf{K}) \to \cdots$$

It is possible to construct a unified framework that will include both of these exact sequences, but we shall not do so here because it involves numerous further results about simplicial complexes. However, it is important to note that in general one does NOT have a Mayer-Vietoris sequence in singular homology for presentations of a space X as a union of two closed subsets, and *this even fails for compact subsets of the* 2*-sphere.*

Example. Let $P \subset \mathbb{R}^2$ be the Polish circle constructed in `polishcircle.pdf` and `polishcircleA.pdf`, which is the union of the graph of $\sin(1/x)$ for $0 < |x| 1$ and the three closed line segments joining $(0, 1)$ to $(0, -2)$, $(0, -2)$ to $(1, -2)$, and $(1, -2)$ to $(1, \sin 1)$; there is a sketch of P in `polishcircleA.pdf`. By the discussion in the two references, P is a compact arcwise connected subset of the plane, and one can use the same argument as in Proposition 2 and Corollary 3 of `polishcircle.pdf` to prove that if K is compact and locally connected and $h : K \to P$ is continuous, then $h[K]$ lies in a contractible open subset of P and hence $H_q(P) = 0$ if $q \neq 0$ (by arcwise connectedness we have $H_0(\Gamma) \cong \mathbb{Z}$). Now let B be the set of points (x, y) in $\mathbb{R}^2$ satisfying

$$0 \quad \leq x \quad \leq 1 \quad \text{and } \textbf{either}$$

$$-2 \quad \leq \quad y \quad \leq \quad \sin(1/x) \quad \text{if} \quad x \neq 0 \quad \textbf{or} \quad -2 \quad \leq \quad y \quad \leq \quad 1 \quad \text{if} \quad x = 0\ .$$

In the drawing on the first page of `polishcircleA.pdf`, B corresponds to the "closed bounded region whose boundary is P," and it follows immediately that $B = \text{Interior}(B) \cup P$, where the two subsets on the right hand side are disjoint, and that B is the closure of Interior(B). It is straightforward to show that the closed line segment $[0, 1] \times \{-\frac{3}{2}\}$ is a strong deformation retract of B; specifically, the retraction r sends (x, y) to $(x, -\frac{3}{2})$ and the homotopy is given by $t \cdot r(x, y) + (1 - t) \cdot (x, y)$. Therefore we know that the singular homology groups of both P and B are zero in all positive dimensions.

Viewing $\mathbb{R}^2 \subset S^2$ in the usual way, let $A = S^2 - \text{Interior}(B)$; then the observations in the preceding paragraph imply that $A \cap B = P$.

If there was an exact Mayer-Vietoris sequence in singular homology of the form

$$\cdots \to H_q(P) \to H_q(A) \oplus H_q(B) \to H_q(S^2) \to H_{q-1}(P) \cdots$$

then the results of the preceding paragraph would imply that $H_q(A) \cong H_q(S^2)$ for all $q \geq 2$, and in particular that the map $H_2(A) \to H_2(S^2)$ is nontrivial. Now A is a proper subset of S^2, and it is elementary to prove the following result:

LEMMA 9. *If $n > 0$ and A is a proper subset of S^n, then the inclusion map induces the trivial homomorphism from $H_n(A)$ to $H_n(S^n) \cong \mathbb{Z}$.*

Proof of Lemma 9. If $\mathbf{p}$ is a point of S^n that does not lie in A, then the homology map defined by inclusion factors as a composite

$$H_n(A) \to H_n(S^n - \{\mathbf{p}\}) \to H_n(S^n)$$

and this map is trivial because the complement of $\mathbf{p}$ is homeomorphic to $\mathbb{R}^n$ and the n-dimensional homology of the latter is trivial.■

This result and the discussion in the paragraphs preceding the lemma yield a contradiction; the source of this contradiction is our assumption that there is an exact Mayer-Vietoris sequence for $S^2 = A \cup B$, and therefore no such sequence can exist.

WHAT GOES WRONG IN THE EXAMPLE? In order to obtain an exact Mayer-Vietoris sequence for closed subsets, one generally needs an extra condition on the regularity of the inclusion maps. One standard type of condition on the closed subsets is that one can find arbitrarily small open neighborhoods such that the subsets are deformation retracts of these neighborhoods. This definitely fails for $P \subset \mathbb{R}^2$. In fact, one can use the methods of `polishcircle.pdf` and `polishcircleA.pdf` to show that P has a cofinal system of decreasing open neighborhoods $\{W_m\}$ such that $W_{m+1} \subset W_m$ is a homotopy equivalence for all m and each neighborhood is homotopy equivalent to S^1. Since $H_1(P) = 0$, there cannot be arbitrarily small open neighborhoods $V \supset P$ such that P is a deformation retract of V (if, say, $V \subset W_1$ and we choose n such that $W_n \subset V$, then the nontriviality of $H_1(W_n) \to H_1(W_1)$ implies the nontriviality of $H_1(W_n) \to H_1(V)$ and hence V cannot be contractible).

A more refined analysis yields axiom (E.4).

THEOREM 10. (Naturality of Mayer-Vietoris sequences) *In the setting of Theorem 5, assume we are given a map of triads f from $(X_1; U_1, V_1)$ to $(X_2; U_2, v_2)$. Then there for all integers q there is a commutative ladder as below in which the horizontal lines represent the long exact Mayer-Vietoris sequences of Theorem 5 and the vertical maps are all induced by f:*

$$\begin{array}{ccccccccccc}
\cdots & \to & H_{q+1}(X_1) & \to & H_q(U_1 \cap V_1) & \to & H_q(U_1) \oplus H_q(V_1) & \to & H_q(X_1) & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & H_{q+1}(X_2) & \to & H_q(U_2 \cap V_2) & \to & H_q(U_2) \oplus H_q(V_2) & \to & H_q(X_2) & \to & \cdots
\end{array}$$

Proof. For $i = 1, 2$ let $\mathcal{F}(i)$ denote the open covering of X_i by U_i and V_i. Then we have the following commutative diagram of chain complexes whose rows are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & S_*(U_1) \cap S_*(V_1) & \to & S_*(U_1) \oplus S_*(V_1) & \to & S_*^{\mathcal{F}(1)}(X_1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & S_*(U_2) \cap S_*(V_2) & \to & S_*(U_2) \oplus S_*(V_2) & \to & S_*^{\mathcal{F}(2)}(X_2) & \to & 0
\end{array}$$

The theorem follows by taking the long exact commutative ladder associated to this diagram.■

For the sake of completeness, we note that our work thus far yields the following conclusion, which corresponds to one of the axioms for a **simplicial** homology theory.

THEOREM 11. *Suppose that the pair (X, A) is obtained by regularly attaching a k-cell to A, and let $D \subset X$ denote the image $f[D^k]$, and let $S \subset X$ denote the image $f[S^{k-1}]$. Then the* *inclusion of* (D, S) *in* (X, A) *induces isomorphisms of singular homology groups from* $H_*(D, S)$ *to* $H_*(X, A)$.

Proof. In `algtopnotes2012.tex` this statement appeared as Theorem VII.6.1 and was derived as a consequence of axioms (A.1)–(A.5), (B.1)–(B.3), (C.1), (D.1)–(D.4) and (E.1)–(E.4). Since we have shown all of these hold for our construction of singular homology, the proof in the cited reference applies directly to yield the stated result.■

II.4 : Equivalence of simplicial and singular homology

(Hatcher, §§ 2.1 – 2.3)

We now have all the tools we need for verifying axiom (D.1), and as noted before this completes the justification of the applications in Unit VII of `algtopnotes2012.pdf`.

THEOREM 1. *Let* $(X, \mathbf{K})$ *be a simplicial complex, let* $(A, \mathbf{L})$ *determine a subcomplex, and let* $\theta_* : H_*(\mathbf{K}, \mathbf{L}) \to H_*(X, A)$ *be the natural transformation from simplicial to singular homology that was described previously. Then* θ_* *is an isomorphism.*

Proof. The idea is to apply Theorem I.1.8 on natural transformations of homology theories on simplicial complex pairs. In order to do this, we must check that singular homology for simplicial complexes satisfies the five properties $(a)-(e)$ listed shortly before the statement of I.1.8. Property (c), which gives the homology of a finite set, is verified in Proposition IV.1.4, and Properties (a), (b), (d) and (e) — which involve long exact sequences, the homology of a contractible space (more precisely, a simplex), excision for adjoining a single simplex, and the homology of a point — are respectively established in Theorem II.2.2, Corollary II.2.5, Theorem II.3.8, and the discussion following the problem stated after Corollary 1.1.4. Since all these properties hold, Theorem I.1.8 implies that the map θ_* must be an isomorphism for all simplicial complex pairs.■

II.5 : Polyhedral generation, direct limits and uniqueness

(Hatcher: 2.1–2.3, 3.F)

None of the material in this section will be used subsequently in these notes, so the reader may proceed directly to the next unit without loss of continuity. Since the material is optional, there will be less motivation, fewer details, and more reliance on references for topics not covered elsewhere in the course.

Here is one particularly important example to illustrate the preceding sentence: Theorem VI.8.1 in Eilenberg and Steenrod shows that *the restrictions of two singular homology theories are naturally isomorphic on the full subcategory of the underlying space pairs* (P, P') *for simplicial complex pairs* $(P', \mathbf{K}') \subset (P, \mathbf{K})$ (*i.e.*, the mappings are arbitrary continuous maps of pairs and not just subcomplex inclusions of one pair in another), and we shall use this fact without further discussion.

As indicated earlier, the key idea in extending simplicial to singular homology is approximating a space X by continuous maps of polyhedra into X, and axiom (C.3) is basically a formalization of this idea.

Polyhedral generation

This property, which is (C.3) on our list, is definitely less elementary than the ones we have discussed thus far, but for a number of reasons this seems to be the best place to verify it. One reason is that it only figures in proving the uniqueness of singular homology up to isomorphism (something that was never used in Unit VII of `algtopnotes2012.pdf`), and the reader may skip the rest of this section without loss of continuity.

In fact, we shall prove a modified version of (C.3); the reasons for making changes are given below.

THEOREM 1. (Polyhderal generation, slightly weakened) *If (X, A) is a pair of topological spaces, and let $u \in H_q(X, A)$, then there is a simplicial complex pair $(\mathbf{K}, \mathbf{K}')$ with $(P', \mathbf{K}') \subset (P, \mathbf{K})$ and a continuous map of pairs*

$$f : (P, P') \longrightarrow (X, A)$$

such that u is in the image of the map f_* from $H_q(P, P')$ to $H_q(X, A)$. Furthermore, if $(\mathbf{K}, \mathbf{K}')$ is a simplicial complex pair with underlying space pair (P, P') and $v \in H_q(P, P')$ maps trivially to $H_q(X, A)$ under the map f_*, then there is another simplicial complex pair $(Q', \mathbf{L}') \subset (Q, \mathbf{L})$ and continuous functions $h : (P, P') \to (Q, Q')$ and $g : (Q, Q') \to (X, A)$ such that the following hold:

(i) *The composite $g \circ h$ is homotopic to f.*

(ii) *We have $0 = h_*(v) \in H_q(Q, Q')$.*

This property has been well known to most (and perhaps nearly all) mathematicians who have worked extensively with algebraic topology (in particular, it is an immediate consequence of results on geometric realizations of semisimplicial sets; one reference suitable for a course at this level is J. P. May, *Simplicial Objects in Algebraic Topology*, University of Chicago Press, Chicago IL, 1982).

Note. This version of (C.3) is slightly weaker than the one stated in `algtopnotes2012.pdf`, in which the maps $(P, P') \to (Q, Q')$ were required to come from *subcomplex inclusions*. We have made this adjustment because the weaker statement is much easier to verify (for the original version, considerably more information involving simplicial complexes is needed) and the weakened version of (C.3) suffices for proving the uniqueness theorem that we want.

We shall use Hatcher's concept of Δ-complex explicitly in the course of the proof, and we shall also need a few properties of such objects.

LEMMA 2. *A finite Δ-complex in the sense of Hatcher is (compact and) Hausdorff.*

Although this property is dismissed as "obvious" on page 104 of Hatcher, some care seems appropriate because quotient spaces of compact Hausdorff spaces are not necessarily Hausdorff (of course they must be compact), so we shall outline the argument here. Hatcher's complex is constructed by taking a finite disjoint union of simplices and identifying selected subsets of faces with the same dimension. In abstract terms, this construction starts with a compact Hausdorff space X (which is the disjoint union of the simplices) and factors out an equivalence relation $\mathcal{R}$ whose graph in $X \times X$ is a closed subset of the latter (verify this explicitly!). One can then use point set topology to prove that the quotient space is Hausdorff. There is a particularly clear account of the proof in Theorem A.5.4 on page 252 of the following text (note that there are several texts on algebraic topology by the same author in the same series, so the precise title is particularly important here):

> **W. S. Massey.** *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics Vol. 56, *Springer-Verlag, New York NY*, 1977.

Another important fact about Δ-complexes is that they are always homeomorphic to simplicial complexes. In fact, the second barycentric subdivision of the Δ-complex decomposition is always a simplicial decomposition, so one can actually say slightly more:

> *If $(\mathbf{K}, \mathbf{K}')$ is a Δ-complex pair with underlying space pair (P, P'), then the second barycentric subdivision induces a simplicial complex structure such that P' corresponds to a simplicial complex.*

This follows from Exercise 2.1.23 on page 133 of Hatcher, and a full proof is given in Theorem 16.41 on pages 148–149 of the following text:

> **B. Gray.** *Homotopy Theory: An Introduction to Algebraic Topology, Pure and Applied Mathematics Vol.* 64. *Academic Press, New York*, 1975.

It will be useful to introduce some notation for iterated faces of a simplex; specifically, if we are given a sequence $\mathbf{i} = (i_1, \cdots, i_r)$ such that $0 \leq i_t \leq q - t + 1$ for all t, the iterated face map $\partial_{\mathbf{i}} : \Delta_{q-r} \to \Delta_q$ will denote the composite of the ordinary face operators $\partial_{i_r} \circ \cdots \circ \partial_{i_1}$.

We now have enough machinery to prove the polyhderal generation property.

Proof of Theorem 1. By construction u is represented on the chain level by a singular chain $y = \sum_j n_j T_j$ such that the coefficients n_j are integers and the maps $T_j : \Delta_q \to X$ are continuous such that $dy \in S_{q-1}(A)$ (which is equivalent to saying that the image of y in $S_q(X, A)$ is a relative cycle). Form a Δ-complex P and a continuous map $g : P \to X$ by starting whose q-simplices σ_j are in 1–1 correspondence with the maps T_j, and identify two $(q - r)$-dimensional faces $\partial_{\mathbf{i}}\sigma \subset \sigma_k$ and $\partial_{\mathbf{j}}\beta \subset \sigma_m$ if $T_k|\partial_{\mathbf{i}}\Delta_q = T_m|\partial_{\mathbf{j}}\Delta_q$. Define g so that its restriction to σ_j is T_j for all j, and let $P' \subset P$ be the Δ-subcomplex of all $(q - 1)$-simplices that have nontrivial coefficients in the absolute chain dy, which by our choice automatically lifts back to $S_{q-1}(A)$. It follows immediately that g passes to a map of pairs, and by the preceding discussion there is a simplicial complex structure on P for which P' is a subcomplex (namely, the second barycentric subdivision). This completes the proof of the first part of the result.

Suppose now that $(\mathbf{K}, \mathbf{K}')$ is a simplicial complex pair with underlying space pair (P, P') and $v \in H_q(P, P')$ maps trivially to $H_q(X, A)$ under the map f_*. If ω is a linear ordering of the vertices of $\mathbf{K}$, then Theorems I.1.7 and II.4.1 imply that v is represented by a relative cycle $z = \sum_j n_j T_j$ in $C_q(\mathbf{K}^\omega)$, where the sum ranges over certain q-simplices of $\mathbf{K}$ that are not in $\mathbf{K}'$ such that $dz \in C_q\big((\mathbf{K}')\big)$. If this cycle goes to zero in $H_q(X, A)$, then there is a singular chain $c = \sum_j m_\ell V_\ell \in S_{q+1}(X)$ such that $dc = z + b$ for some $b \in S_q(A)$; the latter condition implies that b is a linear combination $\sum_j p_n W_n$ for singular simplices $W_n : \Delta_q \to A$. One can now construct a Δ-complex structure $\mathbf{M}$ as in the preceding discussion, and the condition $dc = z + b$ implies that $\mathbf{K}$ is a subcomplex of $\mathbf{M}$; if Q is the underlying space of $\mathbf{M}$, then $P \subset Q$ and the data defining c yield a continuous extension g of f. Let $\mathbf{M}'$ be the union of $\mathbf{K}'$ and the q-simplices in $\mathbf{M}$ corresponding to the singular simplices W_n, let $Q' \subset Q$ be the underlying space of $\mathbf{M}'$, and let $h : (P, P') \to (Q, Q')$ be the inclusion map. By construction $g[Q'] \subset A$ and the map of pairs $f : (P, P') \to (X, A)$ determines an extension to $g : (Q, Q') \to (X, A)$ such that $h_*(v) = 0$, and the final statement in the conclusion of the theorem follows immediately from this.■

We shall merely state what we need and use Chapter VIII of Eilenberg and Steenrod for proofs and other details whenever possible. We shall be working with *quasi-ordered sets* $(D, \leq)$ which satisfy the reflexive and transitive conditions for partial orderings (the symmetric condition $a \leq b$ and $b \leq a \Rightarrow a = b$ is dropped); as in the partially ordered case, a quasi-ordered set defines a category whose objects are the elements of the set, and if $d_1, d_2 \in D$ there is one morphism $d_1 \to d_2$ if $d_1 \leq d_2$ and there are no morphisms otherwise. A quasi-ordered sets $(D, \leq)$ is said to be **directed** if it satisfies the following condition:

($\star$) *If $x, y \in D$ then there is some $z \in P$ such that $x, y \leq z$.*

Linearly ordered sets and lattices are obvious examples for which this condition holds. We are particularly interested in the following special case and certain constructions involving it:

Example. Let $\mathbb{R}^\infty$ be the set of all infinite sequences of real numbers $(x_1, x_2, \cdots)$ such that all but finitely many x_k are zero, and consider the set of $\mathbb{P}$ all simplicial complexes $(P, \mathbf{K})$ in $\mathbb{R}^\infty$ such that the vertices of each simplex are unit vectors (a single nonzero coordinate, which is equal to 1); by finiteness each subspace P of this type lies in some $\mathbb{R}^N \subset \mathbb{R}^\infty$ given by all sequences such that $x_k = 0$ for $k > N$. Condition ($\star$) holds because the union of two such complexes contains both of them. Note that *every simplicial complex is isomorphic to a complex in* $\mathbb{P}$.

Definition. A *directed system* $\{B_x : x \in D\}$ in a category $\mathbb{A}$ is a covariant functor B from the category defined by $(D, \leq)$ to $\mathbb{A}$, and a morphism of directed systems in $\mathbb{A}$, from $\{B_x : x \in D\}$ to $\{B'_x : x \in D'\}$, is a natural transformation of functors.

The simplest way to motivate the concept of direct limit is to look a simple class of examples. Suppose that we have an increasing sequence $\{G_n\}$ of groups (*i.e.*, G_n is a subgroup of G_{n+1} for all positive integers n). Then it is fairly easy to form a limiting object G_∞ which is essentially a monotone union of the groups G_n. More generally, if may view an object L in $\mathbb{A}$ as a directed system $\{\bullet\}(L)$ defined on the category $\{\bullet\}$ with a single morphism (and a single object), then we may define direct limits as follows:

Definition. Given a directed system $B : D \to \mathbb{A}$, a natural transformation $\varphi : B \to \{\bullet\}(L)$ is a *direct limit* if it has the universal mapping property: If $\omega : B \to \{\bullet\}(M)$ is another natural transformation, then there is a unique morphism $h : L \to M$ such that $\{\bullet\}(h) \circ \varphi = \omega$.

The usual sort of argument yields the following standard uniqueness and functoriality results:

THEOREM 3. *(i) If $\varphi : B \to \{\bullet\}(L)$ and $\omega : B \to \{\bullet\}(M)$ are direct limits, then there is a unique isomorphism $h : L \to M$ such that $\{\bullet\}(h) \circ \varphi = \omega$.*

(ii) If $\varphi : B \to \{\bullet\}(L)$ and $\omega : C \to \{\bullet\}(M)$ are direct limits with values in the same category and $F : B \to C$ is a map of directed systems, then there is a unique map of direct limit objects $f_\infty : L \to M$ such that $\{\bullet\}(f_\infty) \circ \varphi = \omega \circ F$. Furthermore, the construction sending F to f_∞ is (covariantly) functorial.■

The usefulness of the direct limit concept obviously depends upon a reasonable existence statement for such objects. The following can be found in Eilenberg and Steenrod:

THEOREM 4. *If $\mathbb{A}$ is a category of groups with operators (for example, modules over a ring or the category of groups), then every directed system in $\mathbb{A}$ has a direct limit $\varphi : B \to \{\bullet\}(L)$, and it has the following properties:*

(i) Every element of L has the form $\varphi_x(u)$ for some $x \in D$ and some $u \in B_x$.

(ii) if $v \in B_x$ is such that $\varphi_x(v)$ is the trivial element, then there is some $y \geq x$ such that v maps to the trivial element of D_y.■

A (the) direct limit of φ is often denoted by dir lim (B) or similar notation, and the universal map is often denoted by something like dir lim (ω).

There are some clear analogies between the conclusions in Theorems 1 and 4; needless to say, we are going to exploit these similarities.

An isomorphism theorem for singular homology theories

Following Eilenberg and Steenrod, we shall say that a pair of compact Hausdorff spaces (P, P') is $\mathbb{P}-$*triangulable* (it can be triangulated) if it is the underlying space pair for a simplicial complex pair $(\mathbf{K}, \mathbf{K'})$ in $\mathbb{P}$. The restriction to $\mathbb{P}$ is added to obtain a family of bounded cardinality which is large enough to include all isomorphism types of simplicial complex pairs.

Let (X, A) be a pair of topological spaces, and define a directed system $\mathbb{P}(X, A)$ whose elements are given by a $\mathbb{P}-$triangulable pair (P, P') and a continuous map of pairs $f : (P, P') \to (X, A)$. The quasi-ordering on such objects

$$f : (P, P') \to (X, A) \qquad \leq \qquad g : (Q, Q') \to (X, A)$$

(often shortened to $f \leq g$) is given by the existence of a continuous mapping of pairs $h : (P, P') \to (Q, Q')$ such that $f \simeq g \circ h$. The following result shows that $\mathbb{P}(X, A)$ satisfies the required condition $(\star)$ for a directed system:

LEMMA 5. *The quasi-ordered set $\mathbb{P}(X, A)$ is directed.*

Proof. We begin with a general observation about $\mathbb{P}(X, A)$. Suppose that (P_1, P_1') is a pair of subcomplexes in $\mathbb{P}$ which is simplicially isomorphic to (P, P'), and let $J : (P_1, P_1') \to (P, P')$ be a simplicial isomorphism. Then we have $f \leq f \circ J \leq f$ in $\mathbb{P}(X, A)$.

Suppose now that we are given $f : (P, P') \to (X, A)$ and $g : (Q, Q') \to (X, A)$. Clearly we can construct a subcomplex pair (P_1, P_1') which is isomorphic to (P, P') and disjoint from Q (take the vertices of P_1 to be vertices which are not in Q). Let J be a simplicial isomorphism as in the preceding paragraph, and take the map α of pairs from the disjoint union $(P_1, P_1') \amalg (Q, Q') = (P_1 \amalg Q,\ P_1' \amalg Q')$ to (X, A) whose restriction to (P_1, P_1') is $f \circ J$ and whose restriction to (Q, Q') is g. By construction we have $\alpha \geq f \circ J, g$, and since $f \circ J \geq f$ we also have $\alpha \geq f, g$, which is exactly what we needed to prove.■

THEOREM 6. *There is a canonical isomorphism Γ from the direct limit of $\{H_*(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\}$ to $H_*(X, A)$, and it is natural with respect to continuous maps of pairs.*

Proof. For each nonnegative integer q there is a natural transformation

$$\gamma_q : \{H_q(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\} \ \longrightarrow \ \{\bullet\}\big(\, H_q(X, A)\,\big)$$

defined by the homology homomorphisms associated to the continuous mappings $g_\alpha : (P_\alpha, P_\alpha') \to (X, A)$, and by the universal mapping property these yield homomorphisms

$$\text{dir lim}\,(\omega) : \text{dir lim}\,\{H_q(P_\alpha, P_\alpha') : \alpha \in \mathbb{P}\} \ \longrightarrow \ H_q(X, A)\ .$$

Theorems 1 and 4 combine to imply that these homomorphisms are isomorphisms.

Suppose now that we are given a continuous map of pairs $\varphi : (X, A) \to (Y, B)$. Composition with φ defines a map of directed sets $\mathbb{P}(\varphi) : \mathbb{P}(X, A) \to \mathbb{P}(Y, B)$, and this construction is functorial with respect to continuous maps of pairs. If we apply the homology functor H_*, we obtain a map of the corresponding direct systems of abelian groups, and by Theorem 3 we obtain a natural transformation from dir lim $\{H_q(P_\alpha, P'_\alpha) : \alpha \in \mathbb{P}\} to H_q(X, A)$; in this setting naturality is with respect to continuous maps of pairs. By Theorems 1 and 4 this natural mapping is an isomorphism.∎

We now have the machinery we need to prove the following uniqueness theorem:

THEOREM 7. *Suppose that* (h_*, ∂) *and* (h'_*, ∂') *satisfy the following weak versions of the axioms for a singular homology theory:*

- (a) *All the data types except possibly* (T.2) *and* (T.4), *and all the axioms except possibly* (A.6), (D.5), (E.3)–(E.4) *and* (C.3).
- (b) *The weaker version of* (C.3) *corresponding to Theorem* 1.

Then there is a unique natural isomorphism $\lambda : h_* \to h'_*$ *such that for each point* p *the isomorphism* $\lambda_{\{p\}}$ *commutes with the normalization isomorphisms* $h_0(\{p\}) \cong \mathbb{Z}$ *and* $h'_0(\{p\}) \cong \mathbb{Z}$.

Proof. The proof of Theorem 6 is valid for an arbitrary axiomatic singular homology satisfying the conditions in the conclusion of Theorem 1, so the conclusion of Theorem 6 remains valid for an abstract singular homology theory satisfying the hypotheses in the present theorem. In other words, if $k = h$ or h' then $k(X, A)$ is naturally isomorphic to the direct limit of the system $\{k_*(P_\alpha, P'_\alpha)$, where $\alpha \in \mathbb{P}(X, A)$.

For each pair of spaces (X, A) the previously cited uniqueness theorem in Eilenberg and Steenrod yields a natural isomorphism λ of directed systems

$$h_*(P_\alpha, P'_\alpha)_{\alpha \in \mathbb{P}(X,A)} \longrightarrow h'_*(P_\alpha, P'_\alpha)_{t \in \mathbb{P}(X,A)} .$$

The direct limits of these systems are $h_*(X, A)$ and $h'_*(X, A)$ respectively, and therefore one obtains a direct limit isomorphism λ_∞ from $h_*(X, A)$ to $h'_*(X, A)$.

The naturality of this isomorphism follows from Theorem 3.∎