SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 1

Fall 2008

I. Foundational material

I.1 : Basic set theory

Problems from Munkres, § 9, p. 64

2(a)–(c). Find if possible a choice function for each of the collections $\mathcal{V}$ as given in the problem, where $\mathcal{V}$ is a nonempty family of nonempty subsets of the integers or the rational numbers.

SOLUTION.

For each of the first three parts, choose a 1–1 correspondence between the integers or the rationals and the positive integers, and consider the well-orderings that the latter inherit from these maps. For each nonempty subset, define the choice function to be the first element of that subset with respect to the given ordering.■

TECHNICAL FOOTNOTE.

(*This uses material from a graduate level measure theory course.*) In Part (d) of the preceding problem, one cannot find a choice function without assuming something like the Axiom of Choice. The following explanation goes beyond the content of this course but is hopefully illuminating. The first step involves the results from Section I.3 which show that the set of all functions from $\{0, 1\}$ to the nonnegative integers is in 1–1 correspondence with the real numbers. If one could construct a choice function over all nonempty subsets of the real numbers, then among other things one can prove that that there is a subset of the reals which is not Lebesgue measurable without using the Axiom of Choice (see any graduate level book on Lebesgue integration; for example, Section 3.4 of Royden). On the other hand, there are models for set theory in which every subset of the real numbers is Lebesgue measurable (see R. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Annals of Math. (2) **92** (1970), pp. 1–56). — It follows that one cannot expect to have a choice function for arbitrary families of nonempty subsets of the reals unless one makes some extra assumption related to the Axiom of Choice.

5. **(a)** Use the Axiom of Choice to show that if $f : A \to B$ is surjective, then f has a right inverse $h : B \to A$.

SOLUTION.

For each $b \in B$ pick $h(b) \in A$ such that $f(a) = b$; we can find these elements by applying the Axiom of Choice to the family of subsets $f^{-1}[\{b\}]$ because surjectivity implies each of these subsets is nonempty. It follows immediately that $b = f(\,h(b)\,)$.■

(b) Show that if $f : A \to B$ is injective and A is nonempty then f has a left inverse. Is the Axiom of Choice needed?

SOLUTION.

For each $x \in A$ define a function $g_x : B \to A$ whose graph consists of all points of the form

$$\big(f(a),\, a\big) \quad \in \quad B \times A$$

togther with all points of the form (b, x) if b does not lie in the image of A. The injectivity of f implies that this subset is the graph of some function g_x, and by construction we have $g_x \circ f(a) = a$ for all $a \in A$. This does NOT require the Axiom of Choice; for each $x \in A$ we have constructed an EXPLICIT left inverse to f. — On the other hand, if we had simply said that one should pick some element of A for each element of $B - f[A]$, then we WOULD have been using the Axiom of Choice.■

Additional exercise

1. Let X be a set and let $A,\, B \subset X$. The *symmetric difference* $A \oplus B$ is defined by the formula

$$A \oplus B \quad = \quad (A - B) \;\cup\; (B - A)$$

so that $A \oplus B$ consists of all objects in A or B but not both. Prove that $\oplus$ is commutative and associative on the set of all subsets of X, that $A \oplus \emptyset = A$ for all A, that $A \oplus A = \emptyset$ for all A, and that one has the following distributivity relation for $A,\, B,\, C \subset X$:

$$A \cap (B \oplus C) \quad = \quad (A \cap B) \;\oplus\; (A \cap C)$$

SOLUTION.

The commutativity law for $\oplus$ holds because

$$B \oplus A = (B - A) \;\cup\; (A - B)$$

by definition and the commutativity of the set-theoretic union operation. The identity $A \oplus A = \emptyset$ follows because

$$A \oplus A = (A - A) \;\cup\; (A - A) = \emptyset$$

and $A \oplus \emptyset = A$ because

$$A \oplus \emptyset = (A - \emptyset) \;\cup\; (\emptyset - A) = A \;\cup\; \emptyset = A \;.$$

In order to handle the remaining associative and distributive identities it is necessary to write things out explicitly, using the fact that every Boolean expression involving a finite list of subsets can be written as a union of intersections of subsets from the list. It will be useful to introduce some algebraic notation in order to make the necessary manipulations more transparent. Let $X \supset A \cup B \cup C$ denote the complement of $Y \subset X$ by $\widehat{Y}$ (or by $Y^{\frown}$ if Y is some compound algebraic expression), and write $P \cap Q$ simply as PQ. Then the symmetric difference can be rewritten in the form $(A\widehat{B}) \cup (B\widehat{A})$. It then follows that

$$(A \oplus B) \oplus C = \left(A\widehat{B} \cup B\widehat{A}\right) \widehat{C} \,\cup\, C \left(A\widehat{B} \cup B\widehat{A}\right)^{\frown} =$$

$$A\widehat{B}\widehat{C} \cup B\widehat{A}\widehat{C} \cup C\left((\widehat{A} \cup B)(\widehat{B} \cup A)\right) = A\widehat{B}\widehat{C} \cup B\widehat{A}\widehat{C} \cup C\left(\widehat{A}\widehat{B} \cup AB\right) =$$

$$A\widehat{B}\widehat{C}\cup\widehat{A}B\widehat{C}\cup\widehat{A}\widehat{B}C\cup ABC\ .$$

Similarly, we have

$$A\oplus(B\oplus C)=A\left(B\widehat{C}\cup C\widehat{B}\right)^{\frown}\cup\left(B\widehat{C}\cup C\widehat{B}\right)\widehat{A}=$$

$$A\left((\widehat{B}\cup C)(\widehat{C}\cup B)\right)\cup B\widehat{C}\widehat{A}\cup C\widehat{B}\widehat{A}=A(\widehat{B}\widehat{C}\cup BC)\cup B\widehat{C}\widehat{A}\cup C\widehat{B}\widehat{A}=$$

$$A\widehat{B}\widehat{C}\cup\widehat{A}B\widehat{C}\cup\widehat{A}\widehat{B}C\cup ABC\ .$$

This proves the associativity of $\oplus$ because both expressions are equal to the last expression displayed above. The proof for distributivity is similar but shorter (the left side of the desired equation has only one $\oplus$ rather than two, and we only need to deal with monomials of degree 2 rather than 3):

$$A\,(B\oplus C)=A(B\widehat{C}\cup C\widehat{B})=AB\widehat{C}\cup A\widehat{B}C$$

$$AB\oplus AC=(AB)(AC)^{\frown}\cup(AC)(AB)^{\frown}=$$

$$\left(AB(\widehat{A}\cup\widehat{C})\right)\cup\left(AC(\widehat{A}\cup\widehat{B})\right)=AB\widehat{C}\cup A\widehat{B}C$$

Thus we have shown that both of the terms in the distributive law are equal to the same set.■

I.2 : Products, relations and functions

Problem from Munkres, § 4, p. 44

4(a). This is outlined in the course notes.

Additional exercises

1. Let X and Y be sets, suppose that A and C are subsets of X, and suppose that B and D are subsets of Y. Verify the following identities:

(i) $A\times(B\cap D)=(A\times B)\cap(A\times D)$

SOLUTION.

Suppose that (x,y) lies in the left hand side. Then $x\in A$ and $y\in B\cap D$. Since the latter means $y\in B$ and $y\in D$, this means that

$$(x,y)\in(A\times B)\cap(A\times D)\ .$$

Now suppose that (x,y) lies in the set displayed on the previous line. Since $(x,y)\in A\times B$ we have $x\in A$ and $y\in B$, and similarly since $(x,y)\in A\times D$ we have $x\in A$ and $y\in D$. Therefore we have $x\in A$ and $y\in B\cap D$, so that $(x,y)\in A\times(B\cap D)$. Thus every member of the the first set is a member of the second set and vice versa, and therefore the two sets are equal.■

(ii) $A\times(B\cup D)=(A\times B)\cup(A\times D)$

SOLUTION.

Suppose that (x,y) lies in the left hand side. Then $x \in A$ and $y \in B \cup D$. If $y \in B$ then $(x,y) \in A \times B$, and if $y \in D$ then $(x,y) \in A \times D$; in either case we have

$$(x,y) \in (A \times B) \cup (A \times D) \ .$$

Now suppose that (x,y) lies in the set displayed on the previous line. If $(x,y) \in A \times B$ then $x \in A$ and $y \in B$, while if $(x,y) \in A \times D$ then $x \in A$ and $y \in D$. In either case we have $x \in A$ and $y \in B \cup D$, so that $(x,y) \in A \times (B \cap D)$. Thus every member of the the first set is a member of the second set and vice versa, and therefore the two sets are equal.∎

(iii) $A \times (Y - D) = (A \times Y) - (A \times D)$

SOLUTION.

Suppose that (x,y) lies in the left hand side. Then $x \in A$ and $y \in Y - D$. Since $y \in Y$ we have $(x,y) \in A \times Y$, and since $y \notin D$ we have $(x,y) \notin A \times D$. Therefore we have

$$A \times (Y - D) \ \subset \ (A \times Y) - (A \times D) \ .$$

Suppose now that $(x,y) \in (A \times Y) - (A \times D)$. These imply that $x \in A$ and $y \in Y$ but $(x,y) \notin A \times D$; since $x \in A$ the latter can only be true if $y \notin D$. Therefore we have that $x \in A$ and $y \in Y - D$, so that

$$A \times (Y - D) \ \supset \ (A \times Y) - (A \times D) \ .$$

This proves that the two sets are equal.∎

(iv) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

SOLUTION.

Suppose that (x,y) lies in the left hand side. Then we have $x \in A$ and $y \in B$, and we also have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cap C$, while the second and fourth imply $y \in B \cap D$. Therefore $(x,y) \in (A \cap C) \times (B \cap D)$ so that

$$(A \times B) \cap (C \times D) \ \subset \ (A \cap C) \times (B \cap D) \ .$$

Suppose now that (x,y) lies in the set on the right hand side of the displayed equation. Then $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A$ and $y \in B$ we have $(x,y) \in A \times B$, and likewise since $x \in C$ and $y \in D$ we have $(x,y) \in C \times D$, so that

$$(A \times B) \cap (C \times D) \ \supset \ (A \cap C) \times (B \cap D) \ .$$

Therefore the two sets under consideration are equal.∎

(v) $(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D)$

SOLUTION.

Suppose that (x,y) lies in the left hand side. Then either we have $x \in A$ and $y \in B$, or else we have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cup C$, while the second and fourth imply $y \in B \cup D$. Therefore (x,y) is a member of $(A \cup C) \times (B \cup D)$ so that

$$(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D) \ .$$

Supplementary note: To see that the sets are not necessarily equal, consider what happens if $A \cap C = B \cap D = \emptyset$ but all of the four sets A, B, C, D are nonempty. Try drawing a picture in the plane to visualize this.■

(vi) $(X \times Y) - (A \times B) = (X \times (Y - B)) \cup ((X - A) \times Y)$

SOLUTION.

Suppose that (x, y) lies in the left hand side. Then $x \in X$ and $y \in Y$ but $(x, y) \notin A \times B$. The latter means that the statement

$$x \in A \ \textbf{and} \ y \in B$$

is false, which is logically equivalent to the statement

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

If $x \notin A$, then it follows that $(x, y) \in \big((X - A) \times Y\big)$, while if $y \notin B$ then it follows that $(x, y) \in \big(X \times (Y - B)\big)$. Therefore we have

$$\big(X \times Y\big) - \big(A \times B\big) \ \subset \ \big(X \times (Y - B)\big) \cup \big((X - A) \times Y\big) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the containment relation on the displayed line. Then we have $(x, y) \in X \times Y$ and also

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

The latter is logically equivalent to

$$x \in A \ \textbf{and} \ y \in B$$

and this in turn means that $(x, y) \notin A \times B$ and hence proves the reverse inclusion of sets.■

I.3 : Cardinal numbers

Problem from Munkres, § 7, p. 51

4. (a) A real number is said to be algebraic if it satisfies some polynomial equation of positive degree with rational (equivalently, integer) coefficients. Prove that the set of all algebraic numbers is countable; you may use the fact that a polynomial of degree n has at most n distinct roots.

SOLUTION.

Let $\mathbb{Q}[t]$ denote the ring of polynomials with rational coefficients, and for each integer $d > 0$ let $\mathbb{Q}[t]_d$ denote the set of polynomials with degree equal to d. There is a natural identification of $\mathbb{Q}[t]_d$ with the subset of $\mathbb{Q}^{d+1}$ consisting of n-tuples whose last coordinate is nonzero, and therefore $\mathbb{Q}[t]_d$ is countable. Since a countable union of countable sets is countable (Munkres, Theorem 7.5, pp. 48–49), it follows that $\mathbb{Q}[t]$ is also countable.

Given an algebraic number α, there is a unique **monic** rational polynomial $p(t)$ of least (positive) degree such that $p(\alpha) = 0$ (the existence of a polynomial of least degree follows from the well-ordering of the positive integers, and one can find a monic polynomial using division by a positive constant; uniqueness follows because if p_1 and p_2 both satisfy the condition then $p_1 - p_2$ is either zero or a polynomial of lower degree which has α as a root). Let p_α be the polynomial associated to α in this fashion. Then p may be viewed as a function from the set $\mathcal{A}$ of algebraic numbers into $\mathbb{Q}[t]$; if f is an arbitrary element of degree $d \geq 0$, then we know that there are at most d elements of $\mathcal{A}$ that can map to p (and if $f = 0$ the inverse image of $\{f\}$ is empty). Letting $\mathcal{A}_f$ be the inverse image of f, we see that $\mathcal{A} = \cup_f \mathcal{A}_f$, so that the left hand side is a countable union of finite sets and therefore is countable.■

(b) A real number is said to be transcendental if it is not algebraic. Prove that the set of transcendental numbers is uncountable. [As Munkres notes, it is surprisingly hard to determine whether a given number is transcendental.]

SOLUTION.

Since every real number is either algebraic or transcendental but not both, we clearly have

$$2^{\aleph_0} = |\mathbb{R}| = |\text{algebraic}| + |\text{transcendental}| \ .$$

We know that the algebraic numbers are countable, so if the transcendental numbers are also countable the right hand side of this equation reduces to $\aleph_0 + \aleph_0$, which is equal to $\aleph_0$, a contradiction. Therefore the set of transcendental numbers is uncountable (in fact, its cardinality is $2^{\aleph_0}$ but the problem did not ask for us to go any further).■

Problem from Munkres, § 9*, p.* 62

5. **(a)** Use the Axiom of Choice to show that if $f : A \to B$ is surjective then it has a right inverse.

SOLUTION.

For each $b \in B$ let $L_b \subset A$ be the inverse image $f^{-1}(\{b\})$. Using the axiom of choice we can find a function g that assigns to each set L_b a point $g^*(L_b) \in L_b$. Define $g(b) = g^*(L_b)$; by construction we have that $g(b) \in f^{-1}(\{b\})$ so that $f(g(b)) = b$. This means that $f \circ g = \mathrm{id}_B$ and that g is a right inverse to f.■

(b) Show that if $A \neq \emptyset$ and $f : A \to B$ is injective, then f has a left inverse. Is the Axiom of Choice needed here?

SOLUTION.

Given an element $z \in A$ define a map $g_z : B \to A$ as follows: If $b = f(a)$ for some a let $g_z(b) = a$. This definition is unambiguous because there is at most one $a \in A$ such that $f(a) = b$. If b does not lie in the image of f, set $g_z(b) = z$. By definition we then have $g_z(f(a)) = a$ for all $a \in A$, so that $g_z \circ f = \mathrm{id}_A$ and g_z is a left inverse to f. Did this use the Axiom of Choice? No. What we actually showed was that for each point of A there is an associated left inverse. However, if we had simply said, "pick some point $z_0 \in A$ and define g using z_0," then we would have used the Axiom of Choice.■

Problem from Munkres, § 11*, p.* 72

8. As noted in Munkres and the course notes, one standard application of Zorn's Lemma is to show that every vector space has a basis. A possibly infinite set A of vectors in the vector space V is said to be linearly independent if each finite subset is in the sense of elementary linear algebra, and such a set is a basis if every vector in V is a finite linear combination of vectors in A.

(a) If A is linearly independent and $\beta \in V$ does not belong to the subspace of linear combinations of elements of A, prove that $A \cup \{\beta\}$ is linearly independent.

SOLUTION.

Note first that $\beta \notin A$ for otherwise it would be a linear combination of elements in A for trivial reasons.

Suppose the set in question is not linearly independent; then some finite subset C is not linearly independent, and we may as well add β to that subset. It follows that there is a relation

$$x_\beta \beta + \sum_{\gamma \in A \cap C} x_\gamma \gamma = 0$$

where not all of the coefficients x_β or x_γ are equal to zero. In fact, we must have $x_\beta \neq 0$ for otherwise there would be some nontrivial linear dependence relationship in $A \cap C$, contradicting our original assumption on A. However, if $x_\beta \neq 0$ then we can solve for β to express it as a linear combination of the vectors in $A \cap C$, and this contradicts our assumption on β. Therefore the set in question must be linearly independent.■

(b) Show that the collection of all linearly independent subsets of V has a maximal element.

SOLUTION.

Let $\mathbf{X}$ be the partially ordered set of linearly independent subsets of V, with inclusion as the partial ordering. In order to apply Zorn's Lemma we need to know that an arbitrary linearly ordered subset $\mathbf{L} \subset \mathbf{X}$ has an upper bound in in $\mathbf{X}$. Suppose that $\mathbf{L}$ consists of the subsets A_t; it will suffice to show that the union $A = \cup_t A_t$ is linearly independent, for then A will be the desired upper bound.

We need to show that if C is a finite subset of A then C is linearly independent. Since each A_t is linearly independent, it suffices to show that there is some r such that $C \subset A_r$, and we do this by induction on $|C|$. If $|C| = 1$ this is clear because $\alpha \in A$ implies $\alpha \in A_t$ for some t. Suppose we know the result when $|C| = k$, and let $D \subset A$ satisfy $|D| = k+1$. Write $D = D_0 \cup \gamma$ where $\gamma \notin D_0$. Then there is some u such that $D_0 \subset A_u$ and some v such that $\gamma \in A_v$. Since $\mathbf{L}$ is linearly ordered we know that either $A_u \subset A_v$ or vice versa; in either case we know that D is contained in one of the sets A_u or A_v. This completes the inductive step, which in turn implies that A is linearly independent and we can apply Zorn's Lemma.■

(c) Show that V has a basis.

SOLUTION.

Let A be a maximal element of $\mathbf{X}$ whose existence was guaranteed by the preceding step in this exercise. We claim that every vector in V is a linear combination of vectors in A. If this were not the case and β was a vector that could not be expressed in this fashion, then by the first step of the exercise the set $A \cup \{\beta\}$ would be linearly independent, contradicting the maximality of A.■

Additional exercises

1. Show that the set of *countable* subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.

SOLUTION.

Let X be the set in question, and let $Y \subset X$ be the subset of all one point subsets. Since there is a 1–1 correspondence between $\mathbb{R}$ and Y it follows that $2^{\aleph_0} = |\mathbb{R}| = |Y| \leq |X|$. Now write X as a union of subfamilies X_n where $0 \leq n \leq \infty$ such that the cardinality of every set in X_n is n if $n < \infty$ and the cardinality of every set in X_∞ is $\aleph_0$.

Suppose now that $n < \infty$. Then X_n is in 1–1 correspondence with the set of all points $(x_1, \cdots, x_n)$ in $\mathbb{R}^n$ such that $x_1 < \cdots < x_n$ (we are simply putting the points of the subset in order). Therefore $|X_0| = 1$ and $|X_n| \leq 2^{\aleph_0}$ for $1 \leq n < \infty$, and it follows that $\cup_{n<\infty} X_n$ has cardinality at most

$$\aleph_0 \cdot 2^{\aleph_0} \ \leq \ 2^{\aleph_0} \cdot 2^{\aleph_0} \ = \ 2^{\aleph_0} \ .$$

So what can we say about the cardinality of X_∞? Let S be the set of all infinite sequences in $\mathbb{R}$ indexed by the positive integers. For each choice of a 1–1 correspondence between an element of X_∞ and $\mathbb{N}^+$ we obtain an element of S, and if we choose one correspondence for each element we obtain a 1–1 map from X_∞ into S. By definition $|S|$ is equal to $\left(2^{\aleph_0}\right)^{\aleph_0}$, which in turn is equal to $2^{\aleph_0 \times \aleph_0} = 2^{\aleph_0}$; therefore we have $|X_\infty| \leq 2^{\aleph_0}$. Putting everything together we have

$$|X| = |\cup_{n<\infty} \ X_n| + |X_\infty| \leq 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}$$

and since we have already established the reverse inequality it follows that $|X| = 2^{\aleph_0}$ as claimed.■

IMPORTANT FOOTNOTE.

The preceding exercise relies on the generalization of the law of exponents for cardinal numbers

$$\gamma^{\alpha\beta} = \left(\gamma^\alpha\right)^\beta$$

that was stated at the end of Section I.3 of the course notes without proof. For the sake of completeness we shall include a proof.

Choose sets A, B, C so that $|A| = \beta$, $|B| = \alpha$ (note the switch!!) and $|C| = \gamma$, and let $\mathbf{F}(S,T)$ be the set of all (set-theoretic) functions from one set S to another set T. With this terminology the proof of the cardinal number equation reduces to finding a 1–1 correspondence

$$\mathbf{F}(A \times B, C) \longleftrightarrow \mathbf{F}\big(A, \ \mathbf{F}(B, C)\ \big) \ .$$

In other words, we need to construct a 1–1 correspondence between functions $A \times B \to C$ and functions $A \to \mathbf{F}(B,C)$. In the language of category theory this is an example of an *adjoint functor* relationship.

Given $f : A \times B \to C$, construct $f^* : A \to \mathbf{F}(B,C)$ by defining $f^*(a) : B \to C$ using the formula

$$[f^*(a)]\,(b) = f(a,b) \ .$$

This construction is onto, for if we are given $h^* : A \to \mathbf{F}(B,C)$ and we define $f : A \times B \to C$ by the formula

$$f(a,b) = [h(a)]\,(b)$$

then $f^* = h$ by construction; in detail, one needs to check that $f^*(a) = h(a)$ for all $a \in A$, which amounts to checking that $[f^*(a)](b) = [h(a)](b)$ for all a and b — but both sides of this equation are equal to $f(a,b)$. To see that the construction is 1–1, note that $f^* = g^* \iff f^*(a) = g*^(a)$ for all a, which is equivalent to $[f^*(a)](b) = [g^*(a)](b)$ for all a and b, which in turn is equivalent to $f(a,b) = g(a,b)$ for all a and b, which is equivalent to $f = g$. Therefore the construction sending f to f^* is 1–1 onto as required.■

For the record, the other exponential law

$$\left(\beta \cdot \gamma\right)^\alpha = \beta^\alpha \cdot \gamma^\alpha$$

may be verified by first noting that it reduces to finding a 1–1 correspondence between $\mathbf{F}(A, B \times C)$ and

$$\mathbf{F}(A,B) \times \mathbf{F}(A,C) \ .$$

This simply reflects the fact that a function $f : A \to B \times C$ is completely determined by the ordered pair of functions $p_B \circ f$ and $p_C \circ f$ where p_B and p_C are the coordinate projections from $B \times C$ to B and C respectively.■

2. Let α and β be cardinal numbers such that $\alpha < \beta$, and let X be a set such that $|X| = \beta$. Prove that there is a subset A of X such that $|A| = \alpha$.

SOLUTION.

The inequality means that there is a 1–1 mapping j from some set A_0 with cardinality α to a set B with cardinality β. Since the cardinality of X equals β it follows that there is a 1–1 correspondence $f : B \to X$. If we take $A = j(f(A_)))$, then $A \subset X$ and $|A| = \alpha$.■

I.4 : The real number system

Problem from Munkres, § 4, *p.* 35

9.(c) If x and y are real numbers such that $x - y > 1$, show that there is an integer n such that $y < n < x$. [*Hint:* Use the conclusion from part (b).]

SOLUTION.

We shall follow the hint. Part (b) states that if a real number x is not an integer, then there is a unique integer n such that $n < x < n + 1$.

The solution to (c) has two cases depending upon whether or not y is an integer. If $y \in \mathbb{Z}$, then the same is true for $y + 1$ and we have

$$y \quad < \quad y \; + \; 1 \quad < \quad y \; + \; (y - x) \quad = \quad x$$

so that $y + 1$ is an integer with the required properties. Suppose now that $y \notin \mathbb{Z}$, so that there is a unique integer m such that $m < y < m + 1$. We then have

$$x \quad = \quad y \; + \; (x - y) \quad m \; + \; 1$$

so that $y < m + 1 < x$.■

Remark. Of course, the integer n in the conclusion of the preceding exercise is not necessarily unique; for example if we have the stronger inequality $x - y \geq 2$ then there are at least two integers between x and y.

Additional exercise

1. Suppose that : $\mathbb{R} \to \mathbb{R}$ is a set-theoretic function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$; some obvious examples of this type are the functions $f(x) = c \cdot x$ for some fixed real number c. Does it follow that **every** such function f has this form? [*Hint:* Why is $\mathbb{R}$ a vector space over $\mathbb{Q}$? Recall that every vector space over a field has a (possibly infinite) basis by Zorn's Lemma.]

SOLUTION.

The answer is emphatically **NO,** and there are many counterexamples. Let $\{ x_\alpha \}$ be a basis for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. If $f : \mathbb{R} \to \mathbb{R}$ is a $\mathbb{Q}$-linear map, then f satisfies the condition in the problem. Thus it is only necessary to find examples of such maps that are not multiplication by a constant. Since $\mathbb{R}$ is uncountable, a basis for it over the rationals contains infinitely many elements. Pick one element x_0 in the basis, and consider the unique $\mathbb{Q}$-linear transformation f which sends x_0 to itself and all other basis vectors to zero. Then f is nonzero but is neither 1–1 nor onto. In contrast, a mapping of the form $T_c(x) = c \cdot x$ for some fixed real number c is 1–1 and onto if $c \neq 0$ and zero if $c = 0$. Therefore there is no c such that $f = T_c$.∎

Generalization – more difficult to verify. In fact, the cardinality of the set of all mappings f with the given properties is $2^{\mathbf{c}}$, where as usual $\mathbf{c} = 2^{\aleph_0}$ (the same as the cardinality of all maps from $\mathbb{R}$ to itself). To see this, first note that the statement in parentheses follows because the cardinality of the set of all such mappings is

$$\mathbf{c}^{\mathbf{c}} \quad = \quad \left(2^{\aleph_0}\right)^{\mathbf{c}} \quad = \quad 2^{\aleph_0 \cdot \mathbf{c}} \quad = \quad 2^{\mathbf{c}} \ .$$

To prove the converse, we first claim that a basis for $\mathbb{R}$ over $\mathbb{Q}$ must contain $\mathbf{c}$ elements. Note that the definition of vector space basis implies that $\mathbb{R}$ is in 1–1 correspondence with the set of finitely supported functions from B to $\mathbb{Q}$, where B is a basis for $\mathbb{R}$ over $\mathbb{Q}$ and finite support means that the coordinate functions are nonzero for all but finitely many basis elements. Thus if β is the cardinality of B, then the means that the cardinality of $\mathbb{R}$ is β (the details of checking this are left to the reader). For each subset A of B we may define a $\mathbb{Q}$-linear map from $\mathbb{R}$ to itself which sends elements of B to themselves and elements of the difference set $B - A$ to zero. Different subsets determine different mappings, so this shows that the set of all f satisfying the given condition has cardinality at least $2^{\mathbf{c}}$. Since the first part of the proof shows that the cardinality is at most $2^{\mathbf{c}}$, this completes the argument.∎

Remark. By the results of Unit II, if one also assumes that the function f is **continuous,** then the answer becomes **YES.** One can use the material from Unit II to prove this quickly as follows: If $r = f(1)$, then by induction and $f(-x) = -f(x)$ implies that $f(a) = r \cdot a$ for all $a \in \mathbb{Z}$; if $a = p/q \in \mathbb{Q}$, then we have

$$q \cdot f(a) \quad = \quad f(q \cdot a) \quad = \quad f(p) \quad = \quad r\,p\,, \qquad \text{yielding} \qquad f(a) \quad = \quad r \cdot \frac{p}{q}$$

so that $f(a) = r \cdot a$ for all $a \in \mathbb{Q}$. By the results of Section II.4, if f and g are two continuous functions such that $f(a) = g(a)$ for all rational numbers a, then $f = g$. Taking g to be multiplication by r, we conclude that f must also be multiplication by c.∎

2. Let $(X, \leq)$ be a well-ordered set, and let $A \subset X$ be a nonempty subset which has an upper bound in X. Prove that A has a least upper bound in X. [*Hint:* If A has an upper bound, set of upper bounds for A has a least element β.]

SOLUTION.

Follow the hint. If A has an upper bound and β is given as in the hint, then by construction β is an upper bound for A, and it is the least such upper bound.∎

II. Metric and topological spaces

II.1 : Metrics and topologies

Problem from Munkres, § 13, p. 83

3. Given a set X, show that the family $\mathbf{T}_c$ of all subsets A such that $X - A$ is countable or $X - A = X$ defines a topology on X. Determine whether the family $\mathbf{T}_\infty$ of all A such that $X - A$ is infinite or empty or X forms a topology on X.

SOLUTION.

X lies in the family because $X - X = \emptyset$ and the latter is finite, while $\emptyset$ lies in the family because $X - \emptyset = X$. Suppose U_α lies in the family for all $\alpha \in A$. To determine whether their union lies in the family we need to consider the complement of that union, which is

$$X - \bigcup_\alpha U_\alpha \quad = \quad \bigcap_\alpha X - U_\alpha \ .$$

Each of the sets in the intersection on the right hand side is either countable or all of X. If at least one of the sets is countable then the whole intersection is countable, and the only other alternative is if each set is all of X, in which case the intersection is X. In either case the complement satisfies the condition needed for the union to belong to $\mathbf{T}_c$. Suppose now that we have two sets U_1 and U_2 in the family. To decide whether their intersection lies in the family we must again consider the complement of $U_1 \cap U_2$, which is

$$(X - U_1) \cup (X - U_2) \ .$$

If one of the two complements in the union is equal to X, then the union itself is equal to X, while if neither is equal to X then both are countable and hence their union is countable. In either case the complement of $U_1 \cap U_2$ satisfies one of the conditions under which $U_1 \cap U_2$ belongs to $\mathbf{T}_c$.

What about the other family? Certainly $\emptyset$ and X belong to it. What about unions? Suppose that X is an infinite set and that U and V lie in this family. Write $E = X - U$ and $F = X - V$; by assumption each of these subsets is either infinite or empty. Is the same true for their intersection? Of course not! Take X to be the positive integers, let E be all the even numbers and let F be all the prime numbers. Then E and F are infinite but the only number they have in common is 2. — Therefore the family $\mathbf{T}_\infty$ is not necessarily closed under unions and hence it does not necessarily define a topology for X.■

Additional exercises

1. In the integers $\mathbb{Z}$ let p be a fixed prime. For each integer $a > 0$ let $U_a(n) = \{n + kp^a, \text{ some } k \in \mathbb{Z}\}$. Prove that the sets $U_a(n)$ form a basis for some topology on $\mathbb{Z}$. [*Hint:* Let $\nu_p(n)$ denote the largest nonnegative integer k such that p^k divides n and show that

$$\mathbf{d}_p(a,b) \quad = \quad \frac{1}{p^{\nu_p(a-b)}}$$

defines a metric on $\mathbb{Z}$].

SOLUTION.

In the definition of $\mathbf{d}_p$ we tacitly assume that $a \neq b$ and set $\mathbf{d}_p(a,a) = 0$ for all a. The nonnegativity of the function and its vanishing if and only if both variables are equal follow from the construction, as does the symmetry property $\mathbf{d}_p(a,b) = \mathbf{d}_p(b,a)$. The Triangle Inequality takes more insight. There is a special class of metric spaces known as **ultrametric spaces**, for which

$$\mathbf{d}(x,y) \leq \max\{\mathbf{d}(x,z), \mathbf{d}(y,z)\}$$

for all x, y, $z \in X$; the Triangle Inequality is an immediate consequence of this ultrametric inequality.

To establish this for the metric $\mathbf{d}_p$, we may as well assume that $x \neq y$ because if $x = y$ the ultrametric inequality is trivial (the left side is zero and the right is nonnegative). Likewise, we may as well assume that all three of x, y, z are distinct, for otherwise the ultrametric inequality is again a triviality. But suppose that $\mathbf{d}_p(x,y) = p^{-r}$ for some nonnegative integer r. This means that $x - y = p^r q$ where q is not divisible by p. If the ultrametric inequality is false, then p^{-r} is greater than either of either $\mathbf{d}_p(x,z)$ and $\mathbf{d}(y,z)$, which in turn implies that both $x - z$ and $y - z$ are divisible by p^{r+1}. But these two conditions imply that $x - y$ is also divisible by p^{r+1}, which is a contradiction. Therefore the ultrametric inequality holds for $\mathbf{d}_p$.

One curious property of this metric is that it takes only a highly restricted set of values; namely 0 and all fractions of the form p^{-r} where r is a nonnegative integer.■

2. Let $A \subset X$ be closed and let $U \subset A$ be open in A. Let V be any open subset of X with $U \subset V$. Prove that $U \cup (V - A)$ is open in X.

SOLUTION.

Since U is open in A there is an open subset W in X such that $U = W \cap A$, and since $U \subset V$ we even have $U = V \cap U = V \cap W \cap A$. But $V \cap W$ is contained in the union of $U = V \cap W \cap A$ and $V - A$, and thus we have

$$U \cup (V - A) \ \subset\ (V \cap W) \cup (V - A) \ \subset\ (U \cup (V - A)) \cup (V - A) \ \subset\ U \cup (V - A)$$

so that $U \cup (V - A) = (V \cap W) \cup (V - A)$. Since A is closed the set $V - A$ is open, and therefore the set on the right hand side of the preceding equation is also open; of course, this means that the set on the left hand side of the equation is open as well.■

3. Let E be a subset of the topological space X. Prove that every open subset $A \subset E$ is also open in X if and only if E itself is open in X.

SOLUTION.

$(\Longrightarrow)$ If $A = E$ then E is open in itself, and therefore the first condition implies that E is open in X. $(\Longleftarrow)$ If E is any subset of X and A is open in E then $A = U \cap E$ where U is open in X. But we also know that E is open in X, and therefore $A = U \cap E$ is also open in X.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 2

Fall 2008

II. Metric and topological spaces

II.2 : Closed sets and limit points

Problems from Munkres, § 17, *pp.* 101 – 102

2. Show that if A is closed in Y and Y is closed in X then A is closed in X.

SOLUTION.

Since A is closed in Y we can write $A = F \cap Y$ where F is closed in X. Since an intersection fo closed set is closed and Y is closed in X , it follows that $F \cap Y = A$ is also closed in x.■

8. Let A, $B \subset X$, and determine whether the following inclusions hold; if equality fails, determine whether containment one way or the other holds.

(a) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

SOLUTION.

Since $C \subset Y$ implies $\overline{C} \subset \overline{Y}$ it follows that $\overline{A \cap B} \subset \overline{A}$ and $\overline{A \cap B} \subset \overline{B}$, which yields the inclusion

$$\overline{A \cap B} = \overline{A} \cap \overline{B} \ .$$

To see that the inclusion may be proper, take A and B to be the open intervals $(0, 1)$ and $(1, 2)$ in the real line. Then the left hand side is empty but the right hand side is the set $\{1\}$.■

(c) $\overline{A - B} = \overline{A} - \overline{B}$

SOLUTION.

This time we shall first give an example where the first set properly contains the second. Take $A = [-1, 1]$ and $B = \{0\}$. Then the left hand side is A but the right hand side is $A - B$. We shall now show that $\overline{A - B} \supset \overline{A} - \overline{B}$ always holds. Given $x \in \overline{A} - \overline{B}$ we need to show that for eacn open set U containing x the intersection $U \cap (A - B)$ is nonempty. Given such an open set U, the condition $x \notin \overline{B}$ implies that $x \in U - \overline{B}$, which is open. Since $x \in \overline{A}$ it follows that

$$A \cap \left(U - \overline{B}\right) \neq \emptyset$$

and since $U - \overline{B} \subset U - B$ it follows that

$$(A - B) \cap U = A \cap (U - B) \neq \emptyset$$

and hence that $x \in \overline{A - B}$.■

19. If $A \subset X$ then the *boundary* $\mathrm{Bd}(A)$ is defined by the expression

$$\mathrm{Bd}(A) = \overline{A} \cap \overline{X - A} \ .$$

(a) Show that $\mathrm{Int}(A)$ and $\mathrm{Bd}(A)$ are disjoint and their union is the closure of A.

SOLUTION.

The interior of A is the complement of $\overline{X - A}$ while the boundary is contained in the latter, so the intersection is empty.■

(b) Show that $\mathrm{Bd}(A)$ is empty if and only if A is both open and closed.

SOLUTION.

($\Longleftarrow$) If A is open then $X - A$ is closed so that $X - A = \overline{X - A}$, and if A closed then $\overline{A} = A$. Therefore

$$\mathrm{Bd}(A) = \overline{A} \cap \overline{X - A} = A \cap (X - A) = \emptyset \ .■$$

($\Longrightarrow$) The set $\mathrm{Bd}(A)$ is empty if and only if $\overline{A}$ and $\overline{X - A}$ are disjoint. Since the latter contains $X - A$ it follows that $\overline{A}$ and $X - A$ are disjoint. Since their union is X this means that $\overline{A}$ must be contained in A, which implies that A is closed. If one reverses the roles of A and $X - A$ in the preceding two sentences, it follows that $X - A$ is also closed; hence A is both open and closed in X.■

(c) Show that U is open if and only if $\mathrm{Bd}(U) = \overline{U} - U$.

SOLUTION.

By definition the boundary is $\overline{U} \cap \overline{X - U}$.

($\Longrightarrow$) If U is open then $X - U$ is closed and thus equal to its own closure, and therefore the definition of the boundary for U reduces to $\overline{U} \cap (X - U)$, which is equal to $\overline{U} - U$.■

($\Longleftarrow$) We shall show that $X - U$ is closed. By definition $\mathrm{Bd}(U) = \mathrm{Bd}(X - U)$, and therefore $\mathrm{Bd}(X - U) = \overline{U} - U \subset X - U$. On the other hand, by part (a) we also know that $(X - U) \cup \mathrm{Bd}(X - U) = \overline{X - U}$. Since both summands of the left hand side are contained in $X - U$, it also follows that the right hand side is contained in $X - U$, which means that $X - U$ is closed in X.■

(d) If U is open is it true that $U = \mathrm{Int}(\overline{U})$? Justify your answer.

SOLUTION.

No. Take $U = (-1, 0) \cup (0, 1)$ as a subset of $\mathbb{R}$. Then the interior of the closure of U is $(-1, 1)$. However, we always have $U \subset \mathrm{Int}(\overline{U})$ because $U \subset \overline{U} \implies U = \mathrm{Int}(U) \subset \mathrm{Int}(\overline{U})$.■

FOOTNOTE.

Sets which have the property described in **19.(d)** are called *regular open sets.*

20. Find the boundary and interior for each of the following subsets of $\mathbb{R}^2$:
(a) $A = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ y = 0 \ \}$

SOLUTION.

We need to find the closures of A and its complement in $\mathbb{R}^2$. The complement of A is the set of all points whose second coordinate is nonzero. We claim it is open. But if $(x, y) \in \mathbb{R}^2 - A$ then $y \neq 0$ and the set $N_{|y|}\left((x, y) \right) \subset \mathbb{R}^2 - A$. Therefore A is closed. But the closure of the complement of A is all of $\mathbb{R}^2$; one easy way of seeing this is that for all $x \in \mathbb{R}$ we have $\lim_{n\to\infty}(x, 1/n) = (x, 0)$, which means that every point of A is a limit point of $\mathbb{R}^2 - A$. Therefore the boundary of A is equal to $A \cap \mathbb{R}^2 = A$; *i.e.*, every point of A is a boundary point.■

(b) $B = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ x > 0 \ \text{ and } \ y \neq 0 \ \}$

SOLUTION.

Again we need to find the closures of B and its complement. It will probably be very helpful to draw pictures of this set and the sets in the subsequent portions of this problem. The closure of B turns out to be the set of all points where $x \geq 0$ (find sequences converging to all points in this set that are not in B **!**) and the complement is just the complement of B because the latter is open in $\mathbb{R}^2$. Therefore the boundary of B is $\overline{B} - B$, and this consists of all points such that either $x = 0$ or both $x > 0$ and $y = 0$ hold.■

(c) $C = A \cup B$

SOLUTION.

The first two sentences in part **(b)** apply here also. The set C consists of all points such that $x > 0$ or $y = 0$ The closure of this set is the set of all points such that $x \geq 0$ or $y = 0$, the complement of C is the set of all points such that $x \geq 0$ and $y \neq 0$, and the closure of this complement is the set of all points such that $x \leq 0$. The intersection of the two sets will be the set of all point such that $x = 0$ or both $x < 0$ and $y = 0$ hold.■

(d) $D = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ x \notin \mathbb{Q} \ \}$

SOLUTION.

The first two sentences in part **(b)** apply here also. Since every real number is a limit of irrational numbers, it follows that the closure of D is all of $\mathbb{R}^2$. The complement of D consists of all points whose first coordinates are rational, and since every real number is a limit of a sequence of rational numbers it follows that the closure of $\mathbb{R}^2 - D$ is also $\mathbb{R}^2$. Therefore the boundary is $\mathbb{R}^2$.■

(e) $E = \{ \ (x, y) \in \mathbb{R}^2 \ \big| \ 0 < x^2 - y^2 < 1 \ \}$

SOLUTION.

The first two sentences in part **(b)** apply here also. In problems of this sort one expects the boundary to have some relationship to the curves defined by changing the inequalities into equations; for this example the equations are $x^2 - y^2 = 0$ and $x^2 - y^2 = 1$. The first of these is a pair of diagonal lines through the origin that make 45 degree angles with the coordinate axes, and the second is a hyperbola going through $(\pm 1, 0)$ with asymptotes given by the lines $x^2 - y^2 = 0$. The closure of E turns out to be the set of points where $0 \leq x^2 - y^2 \leq 1$, and the closure of its complement is the set of all points where either $x^2 - y^2 \leq 0$ or $x^2 - y^2 \geq 1$. The intersection will then be the set of points where $x^2 - y^2$ is either equal to 0 or 1.

For the sake of completeness, here is a proof of the assertions about closures: Suppose that we have a sequence $\{ (x_n, y_n) \}$ in E and the sequence has a limit $(a, b) \in \mathbb{R}^2$. Then

$$a^2 - b^2 \quad = \quad \lim_{n\to\infty} x_n^2 - y_n^2$$

where for each n the n^{th} term on the right hand side lies in the open interval $(0, 1)$, and therefore the limit value on the right hand side must lie in the closed interval $[0, 1]$. Similarly, suppose that we have sequences in the complement of E that converge to some point (a, b). Since the sequence of numbers $z_n = x_n^2 - y_n^2$ satisfies $z_n \in \mathbb{R} - (0, 1)$ and the latter subset is closed, it also follows that

$a^2 - b^2 \in \mathbb{R} - (0,1)$. This proves that the closures of E and its complement are contained in the sets described in the preceding paragraph.

To complete the proof of the closure assertions we need to verify that every point on the hyperbola or the pair of intersecting lines is a limit point of E. Suppose that we are given a point (a,b) on the hyperbola, and consider the sequence of points

$$(x_n,\, y_n) \quad = \quad \left(a - \frac{\sigma(a)}{n},\, b \right)$$

where $\sigma(x)$ is ± 1 depending on whether a is positive or negative (we know that $|a| > 1$ because $a^2 = 1 + b^2$). If we take $z_n = x_n^2 - y_n^2$ as before then $\lim_{n\to\infty} z_n = 1$ and moreover

$$z_n \quad = \quad \left(a - \frac{\sigma(a)}{n} \right)^2 - b^2 \quad = \quad 1 + \frac{1}{n^2} - \frac{2\, a\, \sigma(a)}{n}$$

where the expression on the right hand side is positive if

$$|a| \quad < \quad \frac{1}{2n}\ .$$

To see that this expression is less than 1 it suffices to note that $|a| = \sigma(a) \cdot a$ and

$$\frac{1}{n^2} - \frac{2\,|a|}{n} \quad \leq \quad \frac{1}{n^2} - \frac{2}{n} \quad < \quad 0$$

for all $n \geq 1$. This verifies that every point on the hyperbola is a limit point of E. — Graphically, we are taking limits along horizontal lines; the reader might want to draw a picture in order to visualize the situation.

Suppose now that (a,b) lies on the pair of intersecting lines, so that $a = \pm b$. How do we construct a sequence in E converging to (a,b)? Once again, we take sequences that live on a fixed horizontal line, but this time we choose

$$(x_n,\, y_n) \quad = \quad \left(a + \frac{\sigma(a)}{n},\, b \right)$$

and note that $z_n = x_n^2 - y_n^2$ is equal to

$$\frac{1}{n^2} \quad + \quad \frac{2\,|a|}{n}$$

which is always positive and is also less than 1 if $n > 2|a| + 1$ (the latter follows because the expression in question is less than $(1 + 2|a|)/n$ by the inequality $n^{-2} < n^{-1}$ for all $n \geq 2$). Thus every point on the pair of intersecting lines also lies in $\mathbf{L}(E)$, and putting everything together we have verified that the closure of E is the set we thought it was.■

(f) $F = \{\ (x,y) \in \mathbb{R}^2 \ \big|\ x \neq 0 \ \text{ and } \ y \leq 1/x\ \}$

SOLUTION.

Geometrically, this is the region underneath either the positive or negative branch of the hyperbola $y = 1/x$ with the x-axis removed, and the branches of the hyperbola are included in the set. We claim that the closure of F is the union of F with the y-axis (*i.e.*, the set of points where

$x = 0$). To see that the y-axis is contained in this set, consider a typical point $(0, y)$ and consider the sequence

$$\left(\frac{1}{n}, y\right)$$

whose limit is $(0, y)$; the terms of this sequence lie in the set F for all $y \geq n$, and therefore the y-axis lies in $\mathbf{L}(F)$. Therefore the closure is at least as large as the set we have described. To prove that it is no larger, we need to show that there are no limit points of F such that $x \neq 0$ and $y > 1/x$. But suppose that we have an infinite sequence in F with terms of the form $(x_n.y_n)$ and limit equal to (a, b), where $a \neq 0$. There are two cases depending upon whether a is positive or negative. Whichever case applies, for all sufficiently large values of n the signs of the terms x_n are equal to the sign of a, so we may as well assume that all terms of the sequence have first coordinates with the same sign as a (drop the first finitely many terms if necessary). If $a > 0$ it follows that $x_n > 0$ and $y_n \leq 1/x_n$, which imply $x_n\, y_n \leq 1$ Taking limits we see that $ab \leq 1$ also holds, so that $b \leq 1/a$. On the other hand, if $a < 0$ then it follows that $x_n < 0$ and $y_n \leq 1/x_n$, which imply $x_n\, y_n \geq 1$ Taking limits we see that $ab \geq 1$ also holds, so that $b \leq 1/a$ holds in this case too.

Now we have to determine the closure of the complement of F; we claim it is the set of all points where either $x = 0$ or $x \neq 0$ and $y \geq 1/x$. By definition it contains all points where $x = 0$ or $y > 1/x$, so we need to show that the hyperbola belongs to the set of limit points and if we have a sequence of points of the form (x_n, y_n) in the complement of F with a limit in the plane, say (a, b), then $b \geq 1/a$. Proving the latter proceeds by the same sort of argument given in the preceding paragraph (it is not reproduced here, but it must be furnished in a complete proof). To see that the hyperbola belongs to the set of limit points take a typical point (a, b) such that $a \neq 0$ and $b = 1/a$ and consider the sequence

$$(x_n, y_n) \quad = \quad \left(a,\ \frac{1}{a} + \frac{1}{n}\right)$$

whose terms all lie in the complement of F and whose limit is (a, b).■

Additional exercises

0. Prove or give a counterexample to the following statement: If U and V are disjoint open subsets of a topological space X, then their closures are also disjoint.

SOLUTION.

Let U and V be the open intervals $(-1, 0)$ and $(0, 1)$ respectively. Then their closures are the closed intervals $[-1, 0]$ and $[0, 1]$ respectively, and the intersection of these two sets is $\{0\}$. This counterexample shows that the statement is false.

1. Give an example to show that in a metric space the closure of an open ε disk about a point is not necessarily the set of all points whose distance from the center is $\leq \varepsilon$.

SOLUTION.

Take any set S with the discrete metric and let $\varepsilon = 1$. Then the set of all points whose distance from some particular $s_0 \in S$ is ≤ 1 is all of S, but the open disk of radius 1 centered at s_0 is just the one point subset $\{\, s_0 \,\}$.■

Definition. A subspace D of a topological space X is *dense* if $\overline{D} = X$; equivalently, it is dense if and only if for every nonempty open subset $U \subset X$ we have $U \cap D \neq \emptyset$.

2. For which spaces is X the only dense subset of itself?

SOLUTION.

If X has the discrete topology then every subset is equal to its own closure (because every subset is closed), so the closure of a proper subset is always proper. Conversely, if X is the only dense subset of itself, then for every proper subset A its closure $\overline{A}$ is also a proper subset. Let $y \in X$ be arbitrary, and apply this to $X - \{y\}$. Then it follows that the latter is equal to its own closure and hence $\{y\}$ is open. Since y is arbitrary, this means that X has the discrete topology.■

3. Let U and V be open dense subsets of X. Prove that $U \cap V$ is dense in X.

SOLUTION.

Given a point $x \in X$ and and open subset W such that $x \in W$, we need to show that the intersection of W and $U \cap V$ is nonempty. Since U is dense we know that $W \cap U \neq \emptyset$; let y be a point in this intersection. Since V is also dense in X we know that

$$(U \cap V) \cap W \quad = \quad V \cap (U \cap W) \quad \neq \emptyset$$

and therefore $U \cap V$ is dense. — You should be able to construct examples in the real line to show that the conclusion is not necessarily true if U and V are not open.■

4. A subspace A of a topological space X is said to be *locally closed* if for each $a \in A$ there is an open neighborhood U of a in X such that $U \cap A$ is closed in U. Prove that A is locally closed if and only if A is the intersection of an open subset and a closet set.

SOLUTION.

($\Longleftarrow$) If $A = E \cap U$ where E is closed and U is open then for each $a \in A$ one can take U itself to be the required open neighborhood of a. ($\Longrightarrow$) Given $a \in A$ let U_a be the open set containing a such that $U_a \cap A$ is closed in U_a. This implies that $U_a - A$ is open in U_a and hence also in X. Let $U = \cup_a U_a$. Then by construction $A \subset U$ and

$$U - A \quad = \quad \bigcup_{\alpha} \, (U_a - A)$$

is open in X. If we take

$$E \quad = \quad X - \overline{U - A}$$

then E is closed in X and $A = U \cap E$ where U is open in X and E is closed in X.■

5. **(a)** Suppose that D is dense in X, and let $A \subset X$. Give an example to show that $A \cap D$ is not necessarily dense in A.

SOLUTION.

Let X be the real numbers, let D be the rational numbers and let $A = X - D$. Then $A \cap D = \emptyset$, which is certainly not dense in X.■

(b) Suppose that $A \subset B \subset X$ and A is dense in B. Prove that A is dense in $\overline{B}$.

SOLUTION.

If $x \in \overline{B}$ and U is an open set containing x, then $U \cap B \neq \emptyset$. Let b be a point in this intersection. Since $b \in U$ and A is dense in B it follows that $A \cap U \neq \emptyset$ also. But this means that A is dense in $\overline{B}$.■

6. Let E be a subset of the topological space X. Prove that every closed subset $A \subset E$ is also closed in X if and only if E itself is closed in X.

SOLUTION.

$(\Longrightarrow)$ If $A = E$ then E is closed in itself, and therefore the first condition implies that E is closed in X. $(\Longleftarrow)$ If E is any subset of X and A is closed in E then $A = U \cap E$ where U is closed in X. But we also know that E is closed in X, and therefore $A = U \cap E$ is also closed in X. (Does all of this sound familiar? The exercise is essentially a copy of an earlier one with with "closed" replacing "open" everywhere.) ∎

7. Given a topological space X and a subset $A \subset X$, explain why the closure of the interior of A does not necessarily contain A.

SOLUTION.

Consider the subset A of $\mathbb{R}$ consisting of $(0,1) \cup \{2\}$. The closure of its interior is $[0,1]$.∎

8. If U is an open subset of X and B is an arbitrary subset of X, prove that $U \cap \overline{B} \subset \overline{U \cap B}$.

SOLUTION.

Suppose $x \in U \cap B \subset U \cap \overline{B}$. Then the inclusion $U \cap V \subset \overline{U \cap B}$ shows that $x \in \overline{U \cap B}$. Since $\overline{B} = B \cup \mathbf{L}(B)$ the resulting set-theoretic identity

$$U \cap \overline{B} \;\; = \;\; (U \cap B) \, \cap \, (U \cap \mathbf{L}(B))$$

implies that we need only verify the inclusion

$$(U \cap \mathbf{L}(B)) \quad \subset \quad \overline{U \cap B}$$

and it will suffice to verify the stronger inclusion statement

$$(U \cap \mathbf{L}(B)) \quad \subset \quad \mathbf{L}(U \cap B)\ .$$

Suppose that $x \in U \cap \mathbf{L}(B)$, and let W be an open subset containing x. Then $W \cap U$ is also an open subset containing X, and since $x \in \mathbf{L}(B)$ we know that

$$\Big(U \cap W \, - \{x\} \Big) \, \cap B \;\; \neq \;\; \emptyset\ .$$

But the expression on the left hand side of this display is equal to

$$\Big(W - \{x\} \Big) \, \cap \, U \, \cap \, B$$

and therefore the latter is nonempty, which shows that $x \in \mathbf{L}(U \cap B)$ as required.∎

9. If X is a topological space, then the *Kuratowski closure axioms* are the following properties of the operation $A \to \mathbf{CL}(A)$ sending $A \subset X$ to its closure $\overline{A}$:

(C1) $A \subset \mathbf{CL}(A)$ for all $A \subset X$.
(C2) $\mathbf{CL}\,(\mathbf{CL}(A)\,) = \mathbf{CL}(A)$
(C3) $\mathbf{CL}(A \cup B) = \mathbf{CL}(A) \cup \mathbf{CL}(B)$ for all $A,\ B \subset X$.
(C4) $\mathbf{CL}(\emptyset) = \emptyset$.

Given an arbitrary set Y and a operation $\mathbf{CL}$ assigning to each subset $B \subset Y$ another subset $\mathbf{CL}(B) \subset Y$ such that $\mathbf{(C1)} - \mathbf{(C4)}$ all hold, prove that there is a unique topology $\mathbf{T}$ on Y such that for all $B \subset Y$, the set $\mathbf{CL}(B)$ is the closure of B with respect to $\mathbf{T}$.

SOLUTION.

In order to define a topological space it is enough to define the family $\mathcal{F}$ of closed subsets that satisfies the standard properties: It contains the empty set and Y, it is closed under taking arbitrary intersections, and it is closed under taking the unions of two subsets. If we are given tne abstract operator $\mathbf{CL}$ as above on the set of all subsets of Y let $\mathcal{F}$ be the family of all subsets A such that $\mathbf{CL}(A) = A$. We need to show that this family satisfies the so-called standard properties mentioned in the second sentence of this paragraph.

The empty set belongs to $\mathcal{F}$ by **(C4)**, and Y does by **(C1)** and the assumption that $\mathbf{CL}(A) \subset Y$ for all $A \subset Y$, which includes the case $A = Y$. If A and B belong to $\mathcal{F}$ then the axioms imply

$$A \cup B \quad = \quad \mathbf{CL}(A) \cup \mathbf{CL}(B) \quad = \quad \mathbf{CL}(A \cup B)$$

(use **(C3)** to derive the second equality).

The only thing left to check is that $\mathcal{F}$ is closed under arbitrary intersections. Let $\mathcal{A}$ be a set and let $\{ A_\alpha \}$ be a family of subsets in $\mathcal{F}$ indexed by all $\alpha \in \mathcal{A}$; by assumption we have $\mathbf{CL}(A_\alpha) = A_\alpha$ for all α, and we need to show that

$$\mathbf{CL}\left(\bigcap_\alpha A_\alpha\right) \quad = \quad \bigcap_\alpha A_\alpha \ .$$

By **(C2)** we know that

$$\mathbf{CL}\left(\bigcap_\alpha \ A_\alpha\right) \quad \subset \quad \bigcap_\alpha \ \mathbf{CL}\,(A_\alpha)$$

and thus we have the chain of set-theoretic inclusions

$$\bigcap_\alpha A_\alpha \quad \subset \quad \mathbf{CL}\left(\bigcap_\alpha \ A_\alpha\right) \quad \subset \quad \bigcap_\alpha \ \mathbf{CL}\,(A_\alpha) \quad = \quad \bigcap_\alpha \ A_\alpha$$

which shows that all sets in the chain of inclusions are equal and hence that if $D = \cap_\alpha \ A_\alpha$, then $D = \mathbf{CL}(D)$.$\blacksquare$

FOOTNOTE.

Exercise 21 on page 102 of Munkres is a classic problem in point set topology that is closely related to the closure operator on subsets of a topological space: Namely, if one starts out with a fixed subset and applies a finite sequence of closure and (set-theoretic) complement operations, then one obtains at most 14 distinct sets, and there are examples of subsets of the real line for which this upper bound is realized. Some hints for working this exercise appear in the following web site:

`http://www.math.ou.edu/~nbrady/teaching/f02-5853/hint21.pdf`

10. Suppose that X is a space such that $\{p\}$ is closed for all $x \in X$ (this includes all metric spaces), and let $A \subset X$. Prove the following statements:

(a) $\mathbf{L}(A)$ is closed in X.

SOLUTION.

It suffices to show that $\mathbf{L}(\mathbf{L}(A)) \subset \mathbf{L}(A)$. Suppose that $x \in \mathbf{L}(\mathbf{L}(A))$. Then for every open set U containing x we have $(U - \{x\}) \cap A \neq \emptyset$, so let y belong to this nonempty intersection. Since one point subsets are closed, it follows that $U - \{x\}$ is an open set containing y, and therefore we must have

$$\Big(U - \{x, y\}\Big) \cap A \neq \emptyset$$

and therefore the sets $U - \{x\}$ and A have a nonempty intersection, so that $x \in \mathbf{L}(A)$.■

(b) For each point $b \in \mathbf{L}(A)$ and open set U containing b, the intersection $U \cap A$ is infinite.

SOLUTION.

If all one point subsets of X are closed, then all finite subsets of X are closed, and hence the complements of all finite subsets of X are open. We shall need this to complete the proof.

Suppose that the conclusion is false; *i.e.*, the set $(U - \{b\}) \cap A$ is finite, say with exactly n elements. If F denotes this finite intersection, then by the preceding paragraph $V = U - F$ is an open set, and since $x \notin F$ we also have $x \in V$. Furthermore, we have $(V - \{x\}) \cap A = \emptyset$; on the other hand, since $b \in \mathbf{L}(A)$ we also know that this intersection is nonempty, so we have a contradiction. The contradiction arose from the assumption that $(U - \{b\}) \cap A$ was finite, so this set must be infinite.■

11. Suppose that X is a set and that $\mathbf{I}$ is an operation on subsets of X such that the following hold:

(i) $\mathbf{I}(X) = X$.
(ii) $\mathbf{I}(A) \subset A$ for all $A \subset X$.
(iii) $\mathbf{I}(\mathbf{I}(A)) = \mathbf{I}(A)$ for all $A \subset X$.
(iv) $\mathbf{I}(A \cap B) = \mathbf{I}(A) \cap \mathbf{I}(B)$.

Prove that there is a unique topology $\mathbf{T}$ on X such that $U \in \mathbf{T}$ if and only if $\mathbf{I}(A) = A$.

SOLUTION.

The first and second conditions respectively imply that X and the empty set both belong to $\mathbf{T}$. Furthermore, the fourth condition implies that the intersection of two sets in $\mathbf{T}$ also belongs to $\mathbf{T}$, so it remains to verify the condition on unions. Suppose that A is a set and $U_\alpha \in \mathbf{T}$ for all $\alpha \in A$. Then we have

$$\bigcap_\alpha U_\alpha \quad = \quad \bigcap_\alpha \mathbf{I}(U_\alpha) \quad \subset \quad \mathbf{I}\left(\bigcap_\alpha U_\alpha\right) \quad \subset \quad \bigcap_\alpha U_\alpha$$

where the third property implies the right hand containment; the chain of inequalities implies that $\cup_\alpha ! U_\alpha$ belongs to $\mathbf{T}$, and therefore it follows that the latter is a topology for X such that a set U is open if and only if $\mathbf{I}(U) = U$.■

12. If X is a topological space and $A \subset X$ then the *exterior* of X, denoted by $\mathrm{Ext}(X)$, is defined to be $X - \overline{A}$. Prove that this construction has the following properties:

(a) $\mathrm{Ext}(A \cup B) = \mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.

SOLUTION.

By definition $\mathrm{Ext}(A \cup B)$ is equal to

$$X - \overline{A \cup B} \quad = \quad X - \left(\overline{A} \cup \overline{B}\right) \quad = \quad \left(X - \overline{A}\right) \cap \left(X - \overline{B}\right)$$

which again by definition is equal to $\mathrm{Ext}(A) \cap \mathrm{Ext}(B)$.■

(b) $\mathrm{Ext}(A) \cap A = \emptyset$.

SOLUTION.

Since $A \subset \overline{A}$ it follows that $X - \overline{A} \subset X - A$ and hence $\mathrm{Ext}(A) \cap A \subset (X - A) \cap A = \emptyset$.■

(c) $\mathrm{Ext}(\emptyset) = X$.

SOLUTION.

The empty set is closed and therefore $\mathrm{Ext}(\emptyset) = X - \emptyset = X$.■

(d) $\mathrm{Ext}(A) \subset \mathrm{Ext}\,(\mathrm{Ext}\,(X - A\,))$.

SOLUTION.

What is the right hand side? It is equal to $X - \overline{B}$ where $B = X - \overline{X - A}$. Note that $B = \mathrm{Int}(A)$. Therefore the right hand side may be rewritten in the form

$$X - \overline{(\mathrm{Int}(A)\,)}\ .$$

We know that $\mathrm{Int}(A) \subset A$ and likewise for their closures, and thus the reverse implication holds for the complements of their closures. But the last containment relation is the one to be proved.■

13. Let A_1 and A_2 be subsets of a topological space X, and let B be a subset of $A_1 \cap A_2$ that is closed in both A_1 and A_2 with respect to the subspace to topologies on each of these sets. Prove that B is closed in $A_1 \cup A_2$.

SOLUTION.

We may write $B = A_i \cap F_i$ where F_i is closed in X. It follows that $B = B \cap F_2 = A_1 \cap F_1 \cap F_2$ and $B = B \cap F_1 = A_2 \cap F_2 \cap F_1 = A_2 \cap F_1 \cap F_2$. Therefore

$$B = B \cup B \quad = \quad \Big(A_1 \cap F_1 \cap F_2\Big) \cup \Big(A_2 \cap F_1 \cap F_2\Big) \quad - \quad \Big(A_1 \cup A_2\Big) \ \cap\ \Big(F_1 \cap F_2\Big)$$

which shows that B is closed in $A_1 \cup A_2$.■

Note that the statement and proof remain valid if “closed” is replaced by “open.”■

14. Suppose that A is a closed subset of a topological space X and B is the closure of $\mathrm{Int}(A)$. Prove that $B \subset A$ and $\mathrm{Int}(B) = \mathrm{Int}(A)$.

SOLUTION.

The first follows because $\mathrm{Int}(A) \subset A$, closure preserves set-theoretic inclusion, and $A = \overline{A}$. To prove the second statement, begin by noting that the first set is contained in the second because $B \subset A$. The reverse inclusion follows because $B = \overline{\mathrm{Int}(A)} \supset \mathrm{Int}(A)$ implies

$$\mathrm{Int}(B) \supset \mathrm{Int}\,(\,\mathrm{Int}(A)\,) = \mathrm{Int}(A)\ .■$$

15. Let X be a topological space and let $A \subset Y \subset X$.

(a) Prove that the interior of A with respect to X is contained in the interior of A with respect to Y.

SOLUTION.

The set $\mathrm{Int}_X(A)$ is an open subset of X and is contained in A, so it is also an open subet of Y that is contained in A. Since $\mathrm{Int}_Y(A)$ is the maximal such subset, it follows that $\mathrm{Int}_X(A) \subset \mathrm{Int}_Y(A)$.■

(b) Prove that the boundary of A with respect to Y is contained in the intersection of Y with the boundary of A with respect to X.

SOLUTION.

It will be convenient to let $\mathrm{CL}_U(B)$ denote the closure of B in U in order to write things out unambiguously.

By definition $\mathrm{Bd}_Y(A)$ is equal to $\mathrm{CL}_Y(A) \cap \mathrm{CL}_Y(Y - A)$, and using the formula $\mathrm{CL}_Y(B) = \mathrm{CL}_X(B) \cap Y$ we may rewrite $\mathrm{Bd}_Y(A)$ as the subset $\mathrm{CL}_X(A) \cap \mathrm{CL}_X(Y - A) \cap Y$. Since $Y - A \subset X - A$ we have $\mathrm{CL}_X(Y - A) \subset \mathrm{CL}_X(X - A)$, and this yields the relation

$$\mathrm{Bd}_Y(A) = \mathrm{CL}_X(A) \cap \mathrm{CL}_X(Y - A) \cap Y \subset \mathrm{CL}_X(A) \cap \mathrm{CL}_X(X - A) = \mathrm{Bd}_X(Y)$$

that was to be established.■

(c) Give examples to show that the inclusions in the preceding two statements may be proper (it suffices to give one example for which both inclusions are proper).

SOLUTION.

One obvious class of examples for **(a)** is given by taking A to be a nonempty subset that is not open and to let $Y = A$. Then the interior of A in X must be a proper subset of A but the interior of A in itself is simply A.

Once again, the best way to find examples where BOTH inclusions are proper is to try drawing a few pictures with pencil and paper. Such drawings lead to many examples, and one of the simplest arises by taking $A = [0, 1] \times \{0\}$, $Y = \mathbb{R} \times \{0\}$ and $X = \mathbb{R}^2$. For this example the interior inclusion becomes $\emptyset \subset (0, 1) \times \{1\}$ and the boundary inclusion becomes $\{0, 1\} \times \{0\} \subset [0, 1] \times \{0\}$. The details of verifying these are left to the reader.■

16. Let X be a topological space and let D be a subspace. A point a is called a *frontier point* of $D \subset X$ if every open set containing a contains at least one point of D and at least one point of $X - D$. Prove that a either belongs to D or is a limit point of D.

SOLUTION.

If a does not belong to D and U is an open set containing a, then $D \cap A = (D - \{a\}) \cap A$, and the condition on A implies that the first, hence also the second, intersection is nonempty. But this is just the definition of the set of limit points of D.■

II.3 : Continuous functions

Problems from Munkres, § 18, pp. 111 – 112

2. Suppose that $f : X \to Y$ is continuous and $A \subset X$. If x is a limit point of A, is $f(x)$ a limit point of $f[A]$?

SOLUTION.

Not necessarily. If f is constant then $f[A]$ has no limit points.■

6. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

SOLUTION.

Let $f(x) = x$ if x is rational and 0 if x is irrational. Then f is continuous at 0 because $|x| < \varepsilon \implies |f(x| < \varepsilon$. We claim that f is not continuous anywhere else. What does it mean in terms of δ and ε for f to be discontinuous at x? For some $\varepsilon > 0$ there is no $\delta > 0$ such that $|t - x| < \delta \implies |f(t) - f(x)| < \varepsilon$. Another way of putting this is that for some ε and all $\delta > 0$ sufficiently small, one can find a point t such that $|t - x| < \delta$ and $|f(t) - f(x)| \geq \varepsilon$.

There are two cases depending upon whether $x \neq 0$ is rational or irrational.

The rational case. Let $\varepsilon = |x|/2$ and suppose that $\delta < |x|$. Then there is an irrational number y such that $|y - x| < \delta$, and $|f(y) - f(x)| = |x| > \varepsilon$. Therefore f is not continuous at x.■

The irrational case. The argument is nearly the same. Let $\varepsilon = |x|/2$ and suppose that $\delta < |x|/4$. Then there is a rational number y such that $|y - x| < \delta$, and $|f(y) - f(x)| = |f(y)| > 3|x|/4 > \varepsilon$. Therefore f is not continuous at x.■

8.(a) [*Only for the special case $X = \mathbb{R}$ where the order topology equals the standard topology.*], Let $f : X \to \mathbb{R}$ be continuous. Show that the set of all points where $f(x) \leq g(x)$ is closed in X.

SOLUTION.

See the first proof of Additional Exercise 1 below.■

9.(c) An indexed family of sets $\{ A_\alpha \}$ is said to be *locally finite* if for each point $x \in X$ there is an open neighborhood that is disjoint from all but finitely many sets in the family.

Suppose that f is a set-theoretic function from X to Y such that for each α the restriction $f|A_\alpha$ is continuous. If each set is closed and the family is locally finite, prove that f is continuous.

SOLUTION.

The idea is to find an open covering by sets U_β such that each restriction $f|U_\beta$ is continuous; the continuity of f will follow immediately from this. Given $x \in X$, let U_x be an open subset containing x that is disjoint from all but finitely many closed subsets in the given family. Let $\alpha(1), \cdots \alpha(k)$ be the indices such that $U_x \cap A_\alpha = \emptyset$ unless $\alpha = \alpha(j)$ for some j. Then the subsets $A_{\alpha(j)} \cap U_x$ form a finite closed covering of the latter, and our assumptions imply that the restriction of f to each of these subsets is continuous. But this implies that the restriction of f to the open subset U_a is also continuous, which is exactly what we wanted to prove.■

Additional exercises

1. Give examples of continuous functions from $\mathbb{R}$ to itself that are neither open nor closed.

SOLUTION.

The easiest examples are those for which the image of $\mathbb{R}$ is neither open nor closed. One example of this sort is

$$f(x) = \frac{x^2}{x^2 + 1}$$

whose image is the half-open interval $(0, 1]$.■

2. Let X be a topological space, and let $f,\ g : X \to \mathbb{R}$ be continuous. Prove that the functions $|f|$, $\max(f, g)$ [whose value at $x \in X$ is the larger of $f(x)$ and $g(x)$] and $\min(f, g)$ [whose value at $x \in X$ is the smaller of $f(x)$ and $g(x)$] are all continuous. [*Hints:* If $h : X \to \mathbb{R}$ is continuous, what can one say about the sets of points where $h = 0$, $h < 0$ and $h > 0$? What happens if we take $h = f - g$?]

FIRST SOLUTION.

First of all, if $f : X \to \mathbb{R}$ is continuous then so is $|f|$ because the latter is the composite of a continuous function (absolute value) with the original continuous function and thus is continuous.

We claim that the set of points where $f \geq g$ is closed in X and likewise for the set where $g \geq f$ (reverse the roles of f and g to get this conclusion). But $f \geq g \Longleftrightarrow f - g \geq 0$, and the latter set is closed because it is the inverse image of the closed subset $[0, \infty)$ under the continuous mapping $f - g$.

Let A and B be the closed subsets where $f \geq g$ and $g \geq f$ respectively. Then the maximum of f and g is defined by f on A and g on B, and since this maximum function is continuous on the subsets in a finite closed covering of X, it follows that the global function (the maximum) is continuous on all of X. Similar considerations work for the minimum of f and g, the main difference being that the latter is equal to g on A and f on B.■

SECOND SOLUTION.

First of all, if $f : X \to \mathbb{R}$ is continuous then so is $|f|$ because the latter is the composite of a continuous function (absolute value) with the original continuous function and thus is continuous. One then has the following formulas for $\max(f, g)$ and $\min(f, g)$ that immediately imply continuity:

$$\begin{aligned} \max(f,g) &= \frac{f+g}{2} + \frac{|f-g|}{2} \\ \min(f,g) &= \frac{f+g}{2} - \frac{|f-g|}{2} \end{aligned}$$

Verification of these formulas is a routine exercises that is left to the reader to fill in; for each formula there are two cases depending upon whether $f(x) \leq g(x)$ or vice versa.■

3. Let $f : X \to Y$ be a set-theoretic mapping of topological spaces.

(a) Prove that f is open if and only if $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f[A]\,)$ for all $A \subset X$ and that f is closed if and only if $\overline{f[A]} \subset f[\,\overline{A}\,]$ for all $A \subset X$.

SOLUTION.

Suppose that f is open. Then $\mathrm{Int}(A) \subset A$ implies that $f[\,\mathrm{Int}(A)\,]$ is an open set contained in $f[A]$; since $\mathrm{Int}(\,f[A]\,)$ is the largest such set it follows that $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f[A]\,)$.■

Conversely, if the latter holds for all A, then it holds for all open subsets U and reduces to $f[U] \subset \mathrm{Int}(\,f[U]\,)$. Since the other inclusion also holds (every set contains its interior), it follows that the two sets are equal and hence that $f[U]$ is open in Y.■

Suppose now that f is closed. Then $A \subset \overline{A}$ implies that $f[A] \subset f[\overline{A}]$, so that the latter is a closed subset containing $f[A]$. Since $\overline{f[A]}$ is the smallest such set, it follows that $\overline{f[A]} \subset f[\,\overline{A}\,]$.■

Conversely, if the latter holds for all A, then it holds for all closed subsets F and reduces to $\overline{f[F]} \subset f[F]$. Once again the other inclusion also holds (each set is contained in its closure), and therefore the two sets are equal and $f[F]$ is closed in Y.■

(b) Using this and other results from the course notes, prove that f is closed if and only if $\overline{f[A]} = f[\,\overline{A}\,]$ for all $A \subset X$ and f is continuous and open if and only if $f^{-1}\,[\,\mathrm{Int}(B)\,] = \mathrm{Int}(\,f^{-1}[B]\,)$ for all $B \subset Y$.

SOLUTION.

To see the statement about continuous and closed maps, note that if f is continuous then for all $A \subset X$ we have $f[\,\overline{A}\,] \subset \overline{f[A]}$ (this is the third characterization of continuity in the course notes), while if f is closed then we have the reverse inclusion. This proves the ($\Longrightarrow$) direction. To prove the reverse implication, split the set-theoretic equality into the two containment relations given in the first sentence of this paragraph. One of the containment relations implies that f is continuous and the other implies that f is closed.■

To see the statement about continuous and open maps, note that f is continuous if and only if for all $B \subset Y$ we have $f^{-1}\left[\,\mathrm{Int}(B)\,\right] \subset \mathrm{Int}\left(f^{-1}[B]\right)$ (this is the sixth characterization of continuity in the course notes). Therefore it will suffice to show that f is open if and only if the reverse inclusion holds for all $B \subset Y$. Suppose that f is open and $B \subset Y$. Then by our characterization of open mappings we have

$$f\left(\,\mathrm{Int}\left(f^{-1}[B]\right)\,\right) \subset \mathrm{Int} f\left[\,f^{-1}[B]\,\right] \subset \mathrm{Int}(B)$$

and similarly if we take inverse images under f; but the containment of inverse images extends to a longer chain of containments:

$$\mathrm{Int}\left(f^{-1}[B]\right) \quad \subset \quad f^{-1}\left[\,f\left[\mathrm{Int}\left(f^{-1}[B]\right)\right]\,\right] \quad \subset \quad \mathrm{Int}\left(f^{-1}[B]\right) \quad \subset$$

$$f^{-1}\left[f\left[\mathrm{Int}\left(f^{-1}[B]\,\right)\,\right]\,\right] \quad \subset f^{-1}\left[\,\mathrm{Int}(B)\,\right]$$

This proves the ($\Longrightarrow$) implication. What about the other direction? If we set $B = f[A]$ the hypothesis becomes

$$\mathrm{Int}\left(f^{-1}[\,f[A]\,]\right) \quad \subset \quad f^{-1}\left[\mathrm{Int}(\,f[A]\,)\right]$$

and if we take images over f the containment relation is preserved and extends to yield

$$f\left[\,\mathrm{Int}\left(\,f^{-1}[\,f[A]\,]\right)\,\right] \quad \subset \quad f\left[\,f^{-1}\left[\,\mathrm{Int}(\,f[A]\,)\right]\,\right] \quad \subset \quad \mathrm{Int}(\,f[A]\,)\ .$$

Since $A \subset f^{-1}[\,f[A]\,]$ the left hand side of the previous inclusion chain contains $f[\,\mathrm{Int}(A)\,]$, and if one combines this with the inclusion chain the condition $f[\,\mathrm{Int}(A)\,] \subset \mathrm{Int}(\,f]A]\,)$, which characterizes open mappings, is an immediate consequence.■

4. A mapping of topological spaces $f : X \to Y$ is said to be *light* if for each $y \in Y$ the subspace $f^{-1}[\,\{x\}\,]$ inherits the discrete topology (every subset is both open and closed). Prove that the composite of two continuous light mappings is also light (of course, it is also continuous).

SOLUTION.

Suppose that $f : X \to Y$ and $g : Y \to Z$ are both light mappings. For each $z \in Z$ let

$$E_z \quad = \quad \left(g \circ f\right)^{-1}\ \left[\,\{z\}\,\right] \quad = \quad f^{-1}\ \left[\,g^{-1}[\{z\}]\,\right]\ .$$

Likewise, let $F_z = g^{-1}[\{z\}]$. We need to prove that E_z is discrete in the subspace topology; our hypotheses guarantee that F_z is discrete in the subspace topology. Likewise, if we let $H_y = f^{-1}[\{y\}]$, then H_y is also discrete in the subspace topology.

Let $x \in E_z$, and let $y = f(x)$, so that $y \in F_z$. Since F_z is discrete in the subspace topology, the subset $\{y\}$ is both open and closed, and hence one can find an open set $V_y \subset Y$ and a closed set $A_y \subset Y$ such that $V_y \cap F_z = A_y \cap F_z = \{y\}$. If we take inverse images under f and apply standard set-theoretic identities for such subsets, we see that

$$f^{-1}\left[V_y\right] \ \cap \ E_z \quad = \quad f^{-1}\left[A_y\right] \ \cap \ E_z \quad = \quad H_y$$

and by the continuity of f we know that $f^{-1}[V_y] \cap E_z$ and $f^{-1}[A_y] \cap E_z$ are respectively open and closed in E_z. Since each of these intersections is H_y, it follows that the set H_y is open and closed in E_z. As noted at the end of the previous paragraph we also know that $\{x\}$ is open and closed in H_y. Now if $C \subset B \subset D$ such that C is open (*resp., closed*) in B and B is also is open (*resp., closed*) in D, then C is open (*resp., closed*) in D by the standard properties of the subspace topology. Therefore we have shown that $\{x\}$ is both open and closed in the subspace topology for E_z, and since x was arbitrary this means that E_z must be discrete in the subspace topology.■

POSTSCRIPT. If X is a topological space such that one-point subsets are always closed (for example, if X comes from a metric space), then of course F_z and E_z are discrete closed subsets and have no limit points.■

5. If $f(x,y) = (x^2 - y^2)/(x^2 + y^2)$ unless $x = y = 0$ and $f(0,0) = 0$, show that $r : \mathbb{R}^2 \to \mathbb{R}$ is not continuous at $(0,0)$. [*Hint:* Consider the behavior of f on straight lines through the origin.]

SOLUTION.

Consider the line defined by the parametric equations $x(t) = at$, $y(t) = bt$ where a and b are not both zero; the latter is equivalent to saying that $a^2 + b^2 > 0$. The value of the function $f(at, bt)$ for $t \neq 0$ is given by the following formula:

$$f(at, bt) = \frac{a^2t^2 - b^2t^2}{a^2t^2 + b^2t^2} = \frac{a^2 - b^2}{a^2 + b^2}$$

If $a = 0$ or $b = 0$ then this expression reduces to 1, while if $a = b = 1$ this expression is equal to 0. Therefore we know that for every $\delta > 0$ and $t < \delta/4$ the points $(t, 0)$ and (t, t) lie in the open disk of radius δ about the origin and the values of the function at these points are given by $f(t, 0) = 1$ and $f(t, t) = 0$. If the function were continuous at the origin and its value was equal to L, then we would have that $|L|$, $|L - 1| < \varepsilon$ for all $\varepsilon > 0$. No such number exists, and therefore the function cannot be made continuous at the origin.■

6. Let $f(x,y) = 2\,x^2y/(x^4 + y^2)$ unless $x = y = 0$ and $f(0,0) = 0$, and define $\varphi(t) = (t, at)$ and $\psi(t) = (t, t^2)$.

(a) Show that $\lim_{t\to 0} f \circ \varphi(t) = 0$; *i.e.*, f is continuous on every line through the origin.

SOLUTION.

Direct computation shows that $f(t, at)$ is equal to

$$\frac{2\,a\,t^3}{t^4 + a^2t^2} \quad = \quad \frac{2\,a\,t}{t^2 + a^2}$$

and the limit of this expression as $t \to 0$ is zero **provided** $a \neq 0$. Strictly speaking, this is not enough to get the final conclusion, for one also has to analyze the behavior of the function on the x-axis and y-axis. But for the nonzero points of the x-axis one has $f(t, 0) = 0$ and for the nonzero points of the y-axis one has $f(0, t) = 0$.■

(b) Show that $\lim_{t\to 0} f \circ \psi(t) \neq 0$ and give a rigorous argument to explain why this and the preceding part of the exercise imply f is not continuous at $(0,0)$.

SOLUTION.

Once again, we can write out the composite function explicitly:

$$f(t, t^2) = \frac{2\,t^4}{t^4 + t^4} \quad = \quad 1 \qquad (\text{provided} \quad t \neq 0)$$

The limit of this function as $t \to 0$ is clearly 1.

One could give another $\varepsilon - \delta$ proof to show the function is not continuous as in the preceding exercise, but here is another approach by contradiction: Suppose that f is continuous at the origin. Since φ and ψ are continuous functions, it follows that the composites $f \circ \varphi$ and $f \circ \psi$ are continuous at $t = 0$, and that their values at zero are equal to $f(0,0)$. What can we say about the latter? Using $f \circ \varphi$ we compute it out to be zero, but using $f \circ \psi$ it computes out to $+1$. This is a contradiction, and it arises from our assumption that f was continuous at the origin.■

II.4 : Cartesian products

Problems from Munkres, § 18, pp. 111 – 112

10. If $f : A \to B$ and $g : C \to D$ are continuous, prove that the product map $f \times g : A \times C \to B \times D$ is continuous.

SOLUTION.

This is worked out and generalized in the course notes.■

11. Let $F : X \times Y \to Z$ be a set-theoretic function. We say that F is *continuous in each variable separately* if for each $y_0 \in Y$ the map $h : X \to Z$ defined by $h(x) = F(x, y_0)$ is continuous, and for each $x_0 \in X$ the map $k : Y \to Z$ defined by $k(y) = F(x_0, y)$ is continuous. Show that if F is continuous then F is continuous in each variable separately.

SOLUTION.

Consider the maps $A(y_0) : X \to X \times Y$ and $B(x_0) : Y \to X \times Y$ defined by $[A(y_0)](x) = (x, y_0)$ and $[B(x_0)](y) = (x_0, y)$. Each of these maps is continuous because its projection onto one factor is an identity map and its projection onto the other is a constant map. The maps h and k are composites $F \circ A(y_0)$ and $f \circ B(x_0)$ respectively; since all the factors are continuous, it follows that h and k are continuous.■

FOOTNOTE. The next problem in Munkres gives the standard example of a function from $\mathbb{R}^2 \to \mathbb{R}$ that is continuous in each variable separately but not continuous at the origin. See also the solution to Additional Exercise 5 below. was worked out previously.

Additional exercises

1. (*"A product of products is a product."*) Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and let $\mathcal{A} = \cup\{\mathcal{A}_\beta \mid \beta \in \mathcal{B}\}$ be a partition of $\mathcal{A}$. Construct a bijective map of $\prod\{A_\alpha \mid \alpha \in \mathcal{A}\}$ to the set

$$\prod_\beta \{\prod\{A_\alpha \mid \alpha \in \mathcal{A}_\beta\}\ \}\ .$$

If each A_α is a topological space and we are working with product topologies, prove that this bijection is a homeomorphism.

SOLUTION.

The basic idea is to give axioms characterizing cartesian products and to show that they apply in this situation.

LEMMA. *Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of nonempty sets, and suppose that we are given data consisting of a set P and functions $h_\alpha : P \to A_\alpha$ such that for* **EVERY** *collection of data $(S, \{f_\alpha : S \to A_\alpha\})$ there is a unique function $f : S \to P$ such that $h_\alpha \circ f = f_\alpha$ for all α. Then there is a unique $1-1$ correspondence $\Phi : \prod_\alpha A_\alpha \to P$ such that $h_\alpha \circ \Phi$ is the projection from $\prod_\alpha X_\alpha$ onto A_α for all α.*

If we suppose in addition that each A_α is a topological space, that P is a topological space, that the functions h_α are continuous and the unique map f is always continuous, then Φ is a homeomorphism to $\prod_\alpha A_\alpha$ with the product topology.

Sketch of the proof of the lemma. The existence of Φ follows directly from the hypothesis. On the other hand, the data consisting of $\prod_\alpha A_\alpha$ and the coordinate projections π_α also satisfies the given properties. Therefore we have a unique map Ψ going the other way. By the basic conditions the two respective composites $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are completely specified by the maps $\pi_\alpha \circ \Phi \circ \Psi$ and $h_\alpha \circ \Phi \circ \Psi$. Since $\pi_\alpha \circ \Phi = h_\alpha$ and $h_\alpha \circ \Phi = \pi_\alpha$ hold by construction, it follows that $\pi_\alpha \circ \Phi \circ \Psi = \pi_a lpha$ and $h_\alpha \circ \Phi \circ \Psi = h_\alpha$ and by the uniqueness property it follows that both of the composites $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are identity maps. Thus Φ is a 1–1 correspondence.

Suppose now that everything is topologized. What more needs to be said? In the first place, The product set with the product topology has the unique mapping property for continuous maps. This means that both Φ and Ψ are continuous and hence that Φ is a homeomorphism.■

Application to the exercise. We shall work simultaneously with sets and topological spaces, and morphisms between such objects will mean set-theoretic functions or continuous functions in the respective cases.

For each β let $_\beta$ denote the product of objects whose index belongs to $\mathcal{A}_\beta$ and denote its coordinate projections by p_α. The conclusions amount to saying that there is a canonical morphism from $\prod_\beta P_\beta$ to $\prod_\alpha A_\alpha$ that has an inverse morphism. Suppose that we are given morphisms f_α from the same set S to the various sets A_α. If we gather together all the morphisms for indices α lying in a fixed subset $\mathcal{A}_\beta$, then we obtain a unique map $g_\beta : S \to P_\beta$ such that $p_\alpha \circ g_\beta = f_\alpha$ for all $\alpha in \mathcal{A}_\beta$. Let $q_\beta : \prod_\gamma P_\gamma \to P_\beta$ be the coordinate projection. Taking the maps g_β that have been constructed, one obtains a unique map $F : S \to \prod_\beta P_\beta$ such that $q_\beta \circ F = g_\beta$ for all β. By construction we have that $p_\alpha \circ q_\beta \circ F = f_\alpha$ for all α. If there is a unique map with this property, then $\prod_\beta P_\beta$ will be isomorphic to $\prod_\alpha A_\alpha$ by the lemma. But suppose that θ is any map with this property. Once again fix β. Then $p_\alpha \circ q_\beta \circ F = p_\alpha \circ q_\beta \circ \theta = f_\alpha$ for all $\alpha \in \mathcal{A}_\beta$ implies that $q_\beta \circ F = q_\beta \circ \theta$, and since the latter holds for all β it follows that $F = \theta$ as required.■

2. *Non-Hausdorff topology.* In topology courses one is ultimately interested in spaces that are Hausdorff. However, there are contexts in which certain types of non-Hausdorff spaces arise (for example, the Zariski topologies algebraic geometry, which are defined in most textbooks on that subject).

(a) A topological space X is said to be *irreducible* if it cannot be written as a union $X = A \cup B$, where A and B are proper closed subspaces of X. Show that X is irreducible if and only if every pair of nonempty open subsets has a nonempty intersection. Using this, show that an open subset of an irreducible space is irreducible.

SOLUTION.

First part. We shall show that the negations of the two statements in the second sentence are equivalent; *i.e.*, The space X is reducible (not irreducible) if and only if some pair of nonempty open subsets has a nonempty intersection. This follows because X is reducible $\iff$ we can write $X = A \cup B$ where A and B are nonempty proper closed subsets $\iff$ we can find nonempty proper closed subsets A and B such that $(X - A) \cap (X - B) = \emptyset$ $\iff$ we can find nonempty proper open subsets U and V such that $U \cap V = \emptyset$ (take $U = X - A$ and $V = Y - B$).

Second part. The empty set is irreducible (it has no nonempty closed subsets), so suppose that W is a nonempty open subset of X where X is irreducible. But if U and V are nonempty open subspaces of W, then they are also nonempty open subspaces of X, which we know is irreducible. Therefore by the preceding paragraph we have $U \cap V \neq \emptyset$, which in turn implies that W is irreducible (again applying the preceding paragraph).■

(b) Show that every set with the indiscrete topology is irreducible, and every infinite set with the finite complement topology is irreducible.

SOLUTION. For the first part, it suffices to note that a space with an indiscrete topology has no nonempty proper closed subspaces. For the second part, note that if X is an infinite set with the finite complement topology, then the closed proper subsets are precisely the finite subsets of X, and the union of two such subsets is always finite and this is always a proper subset of X. Therefore X cannot be written as the union of two closed proper subsets.■

(c) Show that an irreducible Hausdorff space contains at most one point.

SOLUTION.

If X is a Hausdorff space and $u, v \in X$ then one can find open subsets U and V such that $u \in U$ and $v \in V$ (hence both are nonempty) such that $U \cap V = \emptyset$. Therefore X is not irreducible because it does not satisfy the characterization of such spaces in the first part of **(a)** above.■

3. Let $f : X \to Y$ be a map, and define the *graph* of f to be the set Γ_f of all points $(x, y) \in X \times Y$ such that $y = f(x)$. Prove that the map $x \to (x, f(x))$ is a homeomorphism from X to Γ_f if and only if f is continuous.

SOLUTION.

Let $\gamma : X \to \Gamma_f$ be the set-theoretic map sending x to $f(x)$. We need to prove that f is continuous $\Longleftrightarrow$ γ is a homeomorphism. Let $j : \Gamma_f \to X \times Y$ be the inclusion map.

($\Longrightarrow$) If f is continuous then γ is continuous. By construction it is 1–1 onto, and a continuous inverse is given explicitly by the composite $\pi_X \circ j$ where π_X denotes projection onto X.■

($\Longleftarrow$) If γ is a homeomorphism then f is continuous because it may be written as a composite $\pi_Y \circ j \circ \gamma$ where each factor is already known to be continuous.■

4. Let X be a topological space that is a union of two closed subspaces A and B, where each of A and B is Hausdorff in the subspace topology. Prove that X is Hausdorff.

SOLUTION.

Since A and B are closed in X we know that $A \times A$ and $B \times B$ are closed in $X \times X$. Since A and B are Hausdorff we know that the diagonals Δ_A and Δ_B are closed in $A \times A$ and $B \times B$ respectively. Since "a closed subset of a closed subset is a closed subset" it follows that Δ_A and Δ_B are closed in $X \times X$. Finally $X = A \cup B$ implies that $\Delta_X = \Delta_A \cup \Delta_B$, and since each summand on the right hand side is closed in $X \times X$ it follows that the left hand side is too. But this means that X is Hausdorff.■

EXAMPLE.

Does the same conclusion hold if A and B are open? No. Consider the topology on $\{1, 2, 3\}$ whose open subsets are the empty set and all subsets containing 2. Then both $\{1, 3\}$ and $\{2\}$ are Hausdorff with respect to the respective subspace topologies, but there union — which is X — is not Hausdorff because all open sets contain 2 and thus one cannot find nonempty open subsets that are disjoint.■

5. Let A be some nonempty set, let $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ and $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ be families of topological spaces, and for each $\alpha \in A$ suppose that $f_\alpha : X_\alpha \to Y_\alpha$ is a homeomorphism. Prove that the product map

$$\prod_\alpha f_\alpha \;:\; \prod_\alpha X_\alpha \longrightarrow \prod_\alpha Y_\alpha$$

is also a homeomorphism. [*Hint:* What happens when you take the product of the inverse maps?]

SOLUTION.

For each α let $g_\alpha = f_\alpha^{-1}$. Then we have

$$\prod_\alpha f_\alpha \circ \prod_\alpha g_\alpha \quad = \quad \prod_\alpha (f_\alpha \circ g_\alpha) \quad = \quad \prod_\alpha \mathrm{id}(Y_\alpha) \quad = \quad \mathrm{id}\big(\textstyle\prod_\alpha Y_\alpha\big)$$

and we also have

$$\prod_\alpha g_\alpha \circ \prod_\alpha f_\alpha \quad = \quad \prod_\alpha (g_\alpha \circ f_\alpha) \quad = \quad \prod_\alpha \mathrm{id}(X_\alpha) \quad = \quad \mathrm{id}\big(\textstyle\prod_\alpha X_\alpha\big)$$

so that the product of the inverses $\prod_\alpha g_\alpha$ is an inverse to $\prod_\alpha f_\alpha$.■

6. Let X be a topological space and let $T : X \times X \times X \to X \times X \times X$ be the map that cyclically permutes the coordinates: $T(x, y, z) = (z, x, y)$ Prove that T is a homeomorphism. [*Hint:* What is the test for continuity of a map into a product? Can you write down an explicit formula for the inverse function?]

SOLUTION.

Let π_i be projection onto the i^{th} factor for $i = 1,\, 2\, 3$. The map T is continuous if and only if each $\pi_i \circ T$ is continuous. But by construction we have $\pi_1 \circ T = \pi_3$, $\pi_2 \circ T = \pi_1$, and $\pi_3 \circ T = \pi_2$, and hence T is continuous.

We can solve directly for T^{-1} to obtain the formula $T^{-1}(u, v, w) = (v, w, u)$. We can prove continuity by looking at the projections on the factors as before, but we can also do this by checking that $T^{-1} = T^2$ and thus is continuous as the composite of continuous functions.■

7. Let $\alpha, \beta \in \{1, 2, \infty\}$, and let $|\cdots|_\alpha$ and $|\cdots|_\beta$ be the norms for $\mathbb{R}^n$ that were described in Section II.1.

(a) Explain why there are positive constants m and M (depending upon α and β) such that

$$m \cdot |x|_\beta \quad \le \quad |x|_\alpha \quad \le \quad M \cdot |x|_\beta$$

for all $x \in \mathbb{R}^n$.

SOLUTION.

This is a special case of the estimates relating the $\mathbf{d}_1$, $\mathbf{d}_2$ and $\mathbf{d}_\infty$ metrics.■

(b) Explain why the interior of the closed unit disk with $\mathbf{d}_\alpha$ radius 1 in $\mathbb{R}^n$ is the set of all x such that $|x|_\alpha < 1$ and the frontier is the set of all x such that $|x|_\alpha = 1$.

SOLUTION.

The set of points such that $|x|_\alpha < 1$ is open, and if $|x|_\alpha = 1$ is open and U is an open neighborhood of x, then U contains the points $(1 \pm t) \cdot x$ for all sufficiently small values of $|t|$.■

(c) Prove that there is a homeomorphism h from $\mathbb{R}^n$ to itself such that $|h(x)|_\beta = |x|_\alpha$ for all x. [*Hints:* One can construct this so that $h(x)$ is a positive multiple of x. It is necessary to be a little careful when checking continuity at the origin.]

SOLUTION.

Define $h(x) = |x|_\alpha \cdot |x|_\beta^{-1} \cdot x$ if $X \neq 0$ and $h(0) = 0$. By (a), the mapping h is continuous with respect to the α-norm if and only if if is continuous with respect to the β-norm, and in fact each norm is continuous with respect to the other. It follows immediately that h is continuous if $x \neq 0$. To see continuity at 0, it suffices to check that if $\varepsilon > 0$ then there is some $\delta > 0$ such that $|x|_\alpha < \delta$ implies $|h(x)|_\alpha = |x|_\beta < \varepsilon$. Since $|x|_\beta \leq |x|_\alpha/m$, we can take $\delta = m \cdot \varepsilon$.

To see that h is a homeomorphism, let k be the map constructed by interchanging the roles of α and β in the preceding discussion. By construction k is an inverse function to h, and the preceding argument together with the inequality $|x|_\alpha \leq M \cdot |x|_\beta$ imply that k is continuous.■

(d) Prove that the hypercube $[-1, 1]^n$ is homeomorphic to the unit disk of all points $x \in \mathbb{R}^n$ satisfying $\sum\ x_i^2 \ = \ 1$ such that the frontier of the hypercube is mapped onto the unit sphere.

SOLUTION.

In the preceding discussion we have constructed a homeomorphism which takes the set of all points satisfying $|x|_\beta \leq 1$ to the set defined by $|x|_\alpha \leq 1$, and likewise if the inequality is replaced by equality. If $\beta = \infty$ then the domain is the hypercube and the set of points with $|x|_\beta = 1$ is its frontier, and if $\alpha = 2$ then the codomain is the ordinary unit disk and the set of points where $|x|_\alpha = 1$ is the unit sphere which bounds that disk.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 3

Fall 2008

III. Spaces with special properties

III.1 : Compact spaces – I

Problems from Munkres, § 26, pp. 170 – 172

3. Show that a finite union of compact subspaces of X is compact.

SOLUTION.

Suppose that $A_i \subset X$ is compact for $1 \leq i \leq n$, and suppose that $\mathcal{U}$ is a family of open subsets of X whose union contains $\cup_i A_i$. Then for each i there is a finite subfamily $\mathcal{U}_i$ whose union contains A_i. If we take $\mathcal{U}^*$ to be the union of all these subfamilies then it is finite and its union contains $\cup_i A_i$. Therefore the latter is compact.■

7. If Y is compact, show that the projection $\pi_X : X \times Y \to X$ is a closed map.

SOLUTION.

We need to show that if $F \subset X \times Y$ is closed then $\pi_X[F]$ is closed in X, and as usual it is enough to show that the complement is open. Suppose that $x \notin \pi_X[F]$. The latter implies that $\{x\} \times Y$ is contained in the open subset $X \times Y - F$, and by the Tube Lemma one can find an open set $V_x \subset X$ such that $x \in V$ and $V_x \times Y \subset X \times Y - F$. But this means that the open set $V_x \subset X$ lies in the complement of $\pi_X[F]$, and since one has a conclusion of this sort for each such x it follows that the complement is open as required.■

8. Let $f : X \to Y$ be a set-theoretic map of topological spaces, and assume Y is compact Hausdorff. Prove that f is continuous if and only if its graph Γ_f (defined previously) is a closed subset of $X \times Y$.

SOLUTION.

($\Longrightarrow$) We shall show that the complement of the graph is open, and this does not use the compactness condition on Y (although it does use the Hausdorff property). Suppose we are given (x, y) such that $y \neq x$. Then there are disjoint open sets V and W in Y such that $y \in V$ and $f(x) \in W$. Since $f^{-1}[W]$ is open in X and contains x, there is an open set U containing x such that $f[U] \subset W$. It follows that $U \times V$ is an open subset of $X \times Y$ that is disjoint from Γ_f. Since we have such a subset for each point in the complement of Γ_f it follows that $X \times Y - \Gamma_f$ is open and that Γ_f is closed.■

($\Longleftarrow$) As in a previous exercises let $\gamma : X \to \Gamma_f$ be the graph map and let $j : \Gamma_f \to X \times Y$ be inclusion. General considerations imply that the map $\pi_X \circ j$ is continuous and 1–1 onto, and γ is the associated set-theoretic inverse. If we can prove that $\pi_X \circ j$ is a homeomorphism, then γ will also be a homeomorphism and then f will be continuous by a previous exercise. The map in question will be a homeomorphism if it is closed, and it suffices to check that it is a composite of closed mappings. By hypothesis j is the inclusion of a closed subset and therefore j is closed, and the preceding exercise shows that π_X is closed. Therefore the composite is a homeomorphism as claimed and the mapping f is continuous.■

Problem from Munkres, § 27, pp. 177 – 178

2. Let $A \subset X$ be a nonempty subset of a metric space X.

(a) Show that $\mathbf{d}(x, A) = 0$ if and only if $X \in \overline{A}$.

SOLUTION.

The function of x described in the problem is continuous, so its set of zeros is a closed set. This closed set contains A so it also contains $\overline{A}$. On the other hand if $x \notin \overline{A}$ then there is an $\varepsilon > 0$ such that $N_\varepsilon(x) \subset X - \overline{A}$, and in this case it follows that $\mathbf{d}(x, A) \geq \varepsilon > 0$.■

(b) Show that if A is compact then $\mathbf{d}(x, A) = \mathbf{d}(x, a)$ for some $a \in A$.

SOLUTION.

The function $f(a) = \mathbf{d}(x, a)$ is continuous and $\mathbf{d}(x, A)$ is the greatest lower bound for its set of values. Since A is compact, this greatest lower bound is a minimum value that is realized at some point of A.■

(c) Define the *ε-neighborhood* $U(A, \varepsilon)$ to be the set of all u such that $\mathbf{d}(u, A) < \varepsilon$. Show that this is the union of the neighborhoods $N_\varepsilon(a)$ for all $a \in A$.

SOLUTION.

The union is contained in $U(A, \varepsilon)$ because $\mathbf{d}(x, a) < \varepsilon$ implies $\mathbf{d}(x, A) < \varepsilon$. To prove the reverse inclusion suppose that y is a point such that $\delta = \mathbf{d}(y, A) < \varepsilon$. It then follows that there is some point $a \in A$ such that $\mathbf{d}(y, a) < \varepsilon$ because the greater than the greatest lower bound of all possible distances. The reverse inclusion is an immediate consequence of the existence of such a point a.■

(d) Suppose that A is compact and that U is an open set containing A. Prove that there is an $\varepsilon > 0$ such that $A \subset U(A, \varepsilon)$.

SOLUTION.

Let $F = X - U$ and consider the function $g(a) = \mathbf{d}(a, F)$ for $a \in A$. This is a continuous function and it is always positive because $A \cap F = \emptyset$. Therefore it takes a positive minimum value, say ε. If $y \in A \subset U(A, \varepsilon)$ then $\mathbf{d}(a, y) < \varepsilon \leq \mathbf{d}(a, F)$ implies that $y \notin F$, and therefore $U(A, \varepsilon)$ is contained in the complement of F, which is U.■

(e) Show that the preceding conclusion need not hold if A is not compact.

SOLUTION.

Take X to be all real numbers with positive first coordinate, let A be the points of X satisfying $y = 0$, and let U be the set of all points such that $y < 1/|x|$. Then for every $\varepsilon > 0$ there is a point not in U whose distance from A is less than ε. For example, consider the points $(2n, 1/n)$.■

1. Let X be a compact Hausdorff space, and let $f : X \to X$ be continuous. Define $X_1 = X$ and $X_{n+1} = f(X_n)$, and set $A = \bigcap_n X_n$. Prove that A is a nonempty subset of X and $f[A] = A$.

SOLUTION.

Each of the subsets X_n is compact by an inductive argument, and since X is Hausdorff each one is also closed. Since each set in the sequence contains the next one, the intersection of finitely many sets $X_{k(1)}, \cdots, X_{k(n)}$ in the collection is the set $X_{k(m)}$ where $k(m)$ is the maximum of the $k(i)$. Since X is compact, the Finite Intersection property implies that the intersection A of these sets is nonempty. We need to prove that $f(A) = A$. By construction A is the set of all points that lie in the image of the k-fold composite $\circ^k f$ of f with itself. To see that f maps this set into itself note that if $a = [\circ^k f](x_k)$ for each positive integer k then $f(a) = [\circ^k f](\, f(x_k)\,)$ for each k. To see that f maps this set onto itself, note that $a = [\circ^k f](x_k)$ for each positive integer k implies that

$$a = f\left([\circ^k f](x_{k+1})\right)$$

for each k.■

2. A topological space X is said to be a k-**space** if it satisfies the following condition: *A subset $A \subset X$ is closed if and only if for all compact subsets $K \subset X$, the intersection $A \cap K$ is closed.* It turns out that a large number of the topological spaces one encounters in topology, geometry and analysis are k-spaces (including all metric spaces and compact Hausdorff spaces), and the textbooks by Kelley and Dugundji contain a great deal of information about these k-spaces (another important reference is the following paper by N. E. Steenrod: *A convenient category of topological spaces*, Michigan Mathematical Journal **14** (1967), pp. 133–152).

(a) Prove that if $(X, \mathbf{T})$ is a Hausdorff topological space then there is a unique minimal topology $\mathbf{T}^\kappa$ containing $\mathbf{T}$ such that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a Hausdorff k-space.

SOLUTION. In this situation it is convenient to work with topologies in terms of their closed subsets. Let $\mathcal{F}$ be the family of closed subsets of X associated to $\mathbf{T}$ and let $\mathcal{F}^*$ be the set of all subsets E such that $E \cap C$ is closed in X for every compact subset $C \subset X$. If E belongs to $\mathcal{F}$ then $E \cap C$ is always closed in X because C is closed, so $\mathcal{F} \subset \mathcal{F}^*$. We claim that $\mathcal{F}^*$ defines a topology on X and X is a Hausdorff k-space with respect to this topology.

The empty set and X belong to $\mathcal{F}^*$ because they already belong to $\mathcal{F}$. Suppose that E_α belongs to $\mathcal{F}^*$ for all α; we claim that for each compact subset $C \subset X$ the set $C \cap \bigcap_\alpha E_\alpha$ is $\mathcal{F}$-closed in X. This follows because

$$C \;\cap\; \bigcap_\alpha E_\alpha \;=\; \bigcap_\alpha (C \cap E_\alpha)$$

and all the factors on the right hand side are $\mathcal{F}$-closed (note that they are compact). To conclude the verification that $\mathcal{F}^*$ is a topology, suppose that E_1 and E_2 belong to $\mathcal{F}^*$. Once again let $C \subset X$ be compact, and observe that the set-theoretic equation

$$C \cap (E_1 \cup E_2) \;=\; (C \cap E_1) \;\cup\; (C \cap E_2)$$

implies the right hand side is $\mathcal{F}$-closed if E_1 and E_2 are.

Therefore $\mathcal{F}^*$ defines the closed subspaces of a topological space; let $\mathbf{T}^\kappa$ be the associated family of open sets. It follows immediately that the latter contains $\mathbf{T}$, and one obtains a Hausdorff space by the following elementary observation: *If $(X, \mathbf{T})$ is a Hausdorff topological space and $\mathbf{T}^*$ is a topology for X containing $\mathbf{T}$, then $(X, \mathbf{T}^*)$ is also Hausdorff.* This is true because the disjoint open sets in $\mathbf{T}$ containing a pair of disjoint points are also (disjoint) open subsets with respect to $\mathbf{T}^*$ containing the same respective points.

We now need to show that $\mathcal{K}(X) = (X, \mathbf{T}^\kappa)$ is a k-space and the topology is the unique minimal one that contains $\mathbf{T}$ and has this property. Once again we switch over to using the closed subsets in all the relevant topologies. The most crucial point is that a subset $D \subset X$ is $\mathcal{F}$-compact if and only if it is $\mathcal{F}^*$-compact; by construction the identity map from $[X, \mathcal{F}^*]$ to $[X, \mathcal{F}]$ is continuous (brackets are used to indicate the subset families are the closed sets), so if D is compact with respect to $\mathcal{F}^*$ it image, which is simply itself, must be compact with respect to $\mathcal{F}$. How do we use this? Suppose we are given a subset $B \subset X$ such that $B \cap D$ is $\mathcal{F}^*$-closed for every $\mathcal{F}^*$-compact subset D. Since the latter is also $\mathcal{F}$-compact and the intersection $B \cap D$ is $\mathcal{F}^*$ compact (it is a closed subspace of a compact space), we also know that $B \cap D$ is $\mathcal{F}$-compact and hence $\mathcal{F}$-closed. Therefore it follows that $\mathcal{F}^*$ is a k-space topology. If we are given the closed subsets for any Hausdorff k-space topology $\mathcal{E}$ containing $\mathcal{F}$, then this topology must contain all the closed sets of $\mathcal{F}^*$. Therefore the latter gives the unique minimal k-space topology containing the topology associated to $\mathcal{F}$.∎

(b) Prove that if $f : X \to Y$ is a continuous map of Hausdorff topological spaces, then f is also continuous when viewed as a map from $\mathcal{K}(X) \to \mathcal{K}(Y)$.

SOLUTION.

Suppose that $F \subset Y$ has the property that $F \cap C$ is closed in Y for all compact sets $C \subset Y$. We need to show that $f^{-1}[F] \cap D$ is closed in X for all compact sets $D \subset X$.

If F and D are as above, then $f[D]$ is compact and by the assumption on f we know that

$$f^{-1}\left[\, F \cap f[D]\,\right] = f^{-1}[F] \cap f^{-1}\left[\, f[D]\,\right)$$

is closed in X with respect to the original topology. Since $f^{-1}\left[\, f(D)\,\right)$ contains the closed compact set D we have

$$f^{-1}[F] \cap f^{-1}\left(f[D]\right) \cap D \quad = \quad f^{-1}[F] \cap D$$

and since the left hand side is closed in X the same is true of the right hand side. But this is what we needed to prove.∎

3. *Non-Hausdorff topology revisited.* A topological space X is *noetherian* if every nonempty family of open subsets has a maximal element. This class of spaces is also of interest in algebraic geometry.

(a) (*Ascending Chain Condition*) Show that a space X is noetherian if and only if every increasing sequence of open subsets

$$U_1 \;\subset\; U_2 \;\subset\; \cdots$$

stabilizes; *i.e.*, there is some positive integer N such that $n \geq N$ implies $U_n = U_N$.

SOLUTION.

$(\Longrightarrow)$ The sequence of open subsets has a maximal element; let U_N be this element. Then $n \geq N$ implies $U_N \subset U_n$ by the defining condition on the sequence, but maximality implies the reverse inclusion. Thus $U_N = U_n$ for $n \geq N$.∎

$(\Longleftarrow)$ Suppose that the Chain Condition holds but there is a nonempty family $\mathcal{U}$ of open subsets with no maximal element. If we pick any open set U_1 in this family then there is another open set U_2 in the family that properly contains U_2. Similarly, there is another open subset U_3 in the family that properly contains U_2, and we can inductively construct an ascending chain of open subspaces such that each properly contains the preceding ones. This contradicts the Ascending Chain Condition. Therefore our assumption that $\mathcal{U}$ had no maximal element was incorrect.∎

(b) Show that a space X is noetherian if and only if every open subset is compact.

SOLUTION.

$(\implies)$ Take an open covering $\{U_\alpha\}$ of U for which each open subset in the family is nonempty, and let $\mathcal{W}$ be the set of all finite unions of subsets in the open covering. By definition this family has a maximal element, say W. If $W = U$ then U is compact, so suppose W is properly contained in U. Then if $u \in U - W$ and U_0 is an open set from the open covering that contains u, it will follow that the union $W \cap U_0$ is also a finite union of subsets from the open covering and it properly contains the maximal such set W. This is a contradiction, and it arises from our assumption that W was properly contained in U. Therefore U is compact.■

$(\impliedby)$ We shall show that the Ascending Chain Condition holds. Suppose that we are given an ascending chain

$$U_1 \subset U_2 \subset \cdots$$

and let $W = \cup_n U_n$. By our hypothesis this open set is compact so the open covering $\{U_n\}$ has a finite subcovering consisting of $U_{k(i)}$ for $1 \le i \le m$. If we take N to be the maximum of the $k(i)$'s it follows that $W = U_N$ and $U_n = U_N$ for $n \ge N$.■

(c) Show that a noetherian Hausdorff space is finite (with the discrete topology). [*Hint:* Show that every open subset is closed.]

SOLUTION.

We begin by verifying the statement in the hint. If U is open in a noetherian Hausdorff space X, then U is compact and hence U is also closed (since X is Hausdorff). Since U is Hausdorff, one point subsets are closed and their complements are open, so the complements of one point sets are also closed and the one point subsets are also open. Thus a noetherian Hausdorff space is discrete. On the other hand, an infinite discrete space does not satisfy the Ascending Chain Condition (pick an infinite sequence of distinct points x_k and let U_n be the first n points of the sequence. Therefore a noetherian Hausdorff space must also be finite.■

(d) Show that a subspace of a noetherian space is noetherian.

SOLUTION.

Suppose $Y \subset X$ where X is noetherian. Let $\mathcal{V} = \{V_\alpha\}$ be a nonempty family of open subspaces of Y, write $V_\alpha = U_\alpha \cap Y$ where U_α is open in X, and let $\mathcal{U} = \{U_\alpha\}$. Since X is noetherian, this family has a maximal element U^*, and the intersection $V^* = U^* \cap Y$ will be a maximal element of $\mathcal{V}$.■

III.2 : Complete metric spaces

Problems from Munkres, § 43, pp. 270 − 271

1. Let X be a metric space.

(a) Suppose that for some $\varepsilon > 0$ that every ε-neighborhood of every point has compact closure. Prove that X is complete.

SOLUTION.

Let $\{x_n\}$ be a Cauchy sequence in X and choose M so large that $m, n \ge M$ implies $\mathbf{d}(x_m.x_n) < \varepsilon$. Then all of the terms of the Cauchy sequence except perhaps the first $M - 1$

lie in the closure of $N_\varepsilon(x_M)$, which is compact. Therefore it follows that the sequence has a convergent subsequence $\{ x_{n(k)} \}$. Let y be the limit of this subsequence; we need to show that y is the limit of the entire sequence.

Let $\eta > 0$ be arbitrary, and choose $N_1 \geq M$ such that $m, n \geq N_1$ implies $\mathbf{d}(x_m, x_n) < \eta/2$. Similarly, let $N_2 \geq M$ be such that $n(k) > N_2$ implies $\mathbf{d}(x_{n(k)}, y) < \eta/2$. If we take N to be the larger of N_1 and N_2, and application of the Triangle Inequality shows that $n \leq N$ implies $\mathbf{d}(x_n, y) < \eta$. Therefore y is the limit of the given Cauchy sequence and X is complete.■

(b) Suppose that for each $x \in X$ there is an $\varepsilon > 0$ that $N_\varepsilon(x)$ has compact closure. Show by example that X is not necessarily complete.

SOLUTION.

Take $U \subset \mathbb{R}^2$ to be the set of all points such that $xy < 1$. This is the region "inside" the hyperbolas $y = \pm 1/x$ that contains the origin. It is not closed in $\mathbb{R}^2$ and therefore cannot be complete. However, it is open and just like all open subsets U of $\mathbb{R}^2$ if $x \in X$ and $N_\varepsilon(x) \subset U$ then $N_{\varepsilon/2}(x)$ has compact closure in U.

3. Two metrics $\mathbf{d}$ and $\mathbf{e}$ on the same set X are said to be metrically equivalent if the identity maps from $(X, \mathbf{d})$ to $(X, \mathbf{e})$ and vice versa are uniformly continuous.

(b) Show that if $\mathbf{d}$ and $\mathbf{e}$ are metrically equivalent then X is complete with respect to one metric if and only if it is complete with respect to the other.

SOLUTION.

By the symmetry of the problem it is enough to show that if $(X, \mathbf{d})$ is complete then so is $(X, \mathbf{e})$. Suppose that $\{ x_n \}$ is a Cauchy seuence with respect to $\mathbf{e}$; we claim it is also a Cauchy sequence with respect to $\mathbf{d}$. Let $\varepsilon > 0$, and take $\delta > 0$ such that $\mathbf{e}(u, v) < \delta$ implies $\mathbf{d}(u, v) < \varepsilon$. If we choose M so that $m, n \geq M$ implies $\mathbf{e}(x_n, x_m) < \delta$, then we also have $\mathbf{d}(x_n, x_m) < \varepsilon$. Therefore the original Cauchy sequence with respect to $\mathbf{e}$ is also a Cauchy sequence with respect to $\mathbf{d}$. By completeness this sequence has a limit, say y, with respect to $\mathbf{d}$, and by continuity this point is also the limit of the sequence with respect to $\mathbf{e}$.■

6. A space X is said to be *topologically complete* if it is homeomorphic to a complete metric space.

(a) Show that a closed subspace of a topologically complete space is topologically complete.

SOLUTION.

This follows because a closed subspace of a complete metric space is complete.■

(c) Show that an open subspace of a topologically complete space is topologically complete. [*Hint:* If $U \subset X$ and X is complete with respect to $\mathbf{d}$, define a continuous real valued function on U by the formula $\phi(x) = 1/\mathbf{d}(X, X - U)$ and embed U in $X \times \mathbb{R}$ by the function $f(x) = (x, \phi(x))$.]

SOLUTION.

The image of the function f is the graph of ϕ, and general considerations involving graphs of continuous functions show that f maps U homeomorphically onto its image. If this image is closed in $X \times \mathbb{R}$ then by **(a)** we know that U is topologically complete, so we concentrate on proving that $f(U) \subset X \times \mathbb{R}$ is closed. The latter in turn reduces to proving that the image is closed in the complete subspace $\overline{U} \times \mathbb{R}$, and as usual one can prove this by showing that the complement of $f[U]$ is open. The latter in turn reduces to showing that if $(x, t) \in \overline{U} \times \mathbb{R} - f[U]$ then there is an open subset containing (x, t) that is disjoint from U.

Since ϕ is continuous it follows immediately that the graph of ϕ is closed in the open subset $U \times \mathbb{R}$. Thus the open set $U \times \mathbb{R} - f[U]$ is open in $\overline{U} \times \mathbb{R}$, and it only remains to consider points in $\overline{U} - U \times \mathbb{R}$. Let (x,t) be such a point. In this case one has $\mathbf{d}(x, X - U) = 0$. There are three cases depending on whether t is less than, equal to or greater than 0. In the first case we have that $(x,t) \in \overline{U} \times (-\infty, 0)$ which is open in $\overline{U} \times \mathbb{R}$ and contains no points of $f[U]$ because the second coordinates of points in the latter set are always positive. Suppose now that $t = 0$. Then by continuity of distance functions there is an open set $V \subset \overline{U}$ such that $x \in V$ and $\mathbf{d}(y, X - U) < 1$ for all $y \in V$. It follows that $V \times (-1, 1)$ contains (x,t) and is disjoint from $f[U]$. Finally, suppose that $t > 0$. Then by continuity there is an open set $V \subset \overline{U}$ such that $x \in V$ and

$$\mathbf{d}(y, X - U) \quad < \quad \frac{2}{3t}$$

for all $y \in V$. It follows that $V \times (t/2, 3t/2)$ contains (x,t) and is disjoint from $f[U]$, and this completes the proof of the final case.■

Additional exercises

1. Show that the Nested Intersection Property for complete metric spaces does not necessarily hold for nested sequences of closed subsets $\{ A_n \}$ if $\lim_{n\to\infty} \mathrm{diam}(A_n) \neq 0$; *i.e.*, in such cases one might have $\cap_n A_n = \emptyset$. [*Hint:* Consider the set A_n of all continuous functions f from $[0,1]$ to itself that are zero on $[\frac{1}{n}, 1]$ and also satisfy $f(0) = 1$.]

SOLUTION.

Follow the hint, and try to see what a function in the intersection would look like. In the first place it has to satisfy $f(0) = 1$, but for each $n > 0$ it must be zero for $t \geq 1/n$. The latter means that the $f(t) = 0$ for all $t > 0$. Thus we have determined the values of f everywhere, but the function we obtained is not continuous at zero. Therefore the intersection is empty. Since every function in the set A_n takes values in the closed unit interval, it follows that if f and g belong to A_n then $\|f - g\| \leq 1$ and thus the diameter of A_n is at most 1 for all n. In fact, the diameter is exactly 1 because $f(0) = 1$.

For the sake of completeness, we should note that each set A_n is nonempty. One can construct a "piecewise linear" function in the set that is zero for $t \geq 1/n$ and decreases linearly from the 1 to 0 as t increases from 0 to $1/n$. (Try to draw a picture of the graph of this function!)■

2. Let ℓ^2 be the set of all real sequences $\mathbf{x} = \{ x_n \}$ such that $\sum_n |x_n|^2$ converges. The results of Exercise 10 on page 128 of Munkres show that ℓ^2 is a normed vector space with the norm

$$|\,\mathbf{x}\,| = \left(\sum_n |x_n|^2 \right)^{1/2} .$$

Prove that ℓ^2 is complete with respect to the associated metric. [*Hint:* If $\mathbf{p}_i$ gives the i^{th} term of an element in ℓ^2, show that $\mathbf{p}_i$ takes Cauchy sequences to Cauchy sequences. This gives a candidate for the limit of a Cauchy sequence in ℓ^2. Show that this limit candidate actually lies in ℓ^2 and that the Cauchy sequence converges to it. See also Royden, *Real Analysis*, Section 6.3, pages 123–127, and also Rudin, *Principles of Mathematical Analysis*, Theorem 11.42 on page 329–330 together with the discussion on the following two pages.]

SOLUTION.

For each positive integer k let $H_k(y)$ be the vector whose first k coordinates are the same as those of y and whose remaining coordinates are zero, and let $T_k = I - H_k$ (informally, these are "head" and "tail" functions). Then H_k and T_k are linear transformations and $|H_k(y)|,\ |T_k(y)|\ \leq |y|$ for all y.

Since Cauchy sequences are bounded there is some $B > 0$ such that $|x_n| \leq B$ for all n.

Let x be given as in the hint. We claim that $x \in \ell^2$; by construction we know that $H_k(x) \in \ell^2$ for all k. By the completeness of $\mathbb{R}^M$ we know that $\lim_{n\to\infty} H_k(x_n) = H_k(x)$ and hence there is an integer P such that $n \geq P$ implies $|H_k(x) - H_k(x_n)| < \varepsilon$. If $n \geq P$ we then have that

$$|H_k(x)| \ \leq\ |H_k(x_n)| + |H_k(x) - H_M(k_n)| \ \leq\ |x_n| + |H_M(x) - H_M(x_n)| \ <\ B + \varepsilon\ .$$

By construction $|x|$ is the least upper bound of the numbers $|H_k(x)|$ if the latter are bounded, and we have just shown the latter are bounded. Therefore $x \in \ell^2$; in fact, the argument can be pushed further to show that $|x| \leq B$, but we shall not need this.

We must now show that x is the limit of the Cauchy sequence. Let $\varepsilon > 0$ and choose M this time so that $n,\ m \geq M$ implies $|x_m - x_n| < \varepsilon/6$. Now choose N so that $k \geq N$ implies $|T_k(x_M)| < \varepsilon/6$ and $|T_k(x)| < \varepsilon/3$; this can be done because the sums of the squares of the coordinates for x_M and x are convergent. If $n \geq M$ then it follows that

$$|T_k(x_n)| \ \leq\ |T_k(x_M)| + |T_k(x_n) - T_k(x_M)| =\ |T_k(x_n)| \ \leq\ |T_k(x_M)| + |T_k(x_n - x_M)| \ \leq$$

$$|T_k(x_M)| + |x_n - x_M| \ \leq\ \frac{\varepsilon}{6} + \frac{\varepsilon}{6} \ =\ \frac{\varepsilon}{3}\ .$$

Choose P so that $P \geq M + N$ and $n \geq P$ implies $|H_N(x) - H_N(x_n)| < \frac{\varepsilon}{3}$. If $n \geq P$ we then have

$$|x - x_n| \ \leq\ |H_N(x) - H_N(x_n)| + |T_N(x) - T_N(x_n)| \ \leq$$

$$|H_N(x) - H_N(x_n)| \ +\ |T_N(x)| \ +\ |T_N(x_n)| \ <\ \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \ =\ \varepsilon\ .$$

This completes the proof.■

III.3 : Implications of completeness

Problems from Munkres, § 48, *pp.* 298 – 300

1. Let X be the countable union $\cup_n\ B_n$. Show that if X is a nonempty Baire space then at least one of the sets $\overline{B_n}$ has a nonempty interior.

SOLUTION.

If the interior of the closure of B_n is empty, then B_n is nowhere dense. Thus if the interior of the closure of each B_n is empty then X is of the first category.■

2. Baire's Theorem implies that $\mathbb{R}$ cannot be written as a countable union of closed subsets having nonempty interiors. Show this fails if the subsets are not required to be closed.

SOLUTION.

View the real numbers as a vector space over the rationals. The previous argument on the existence of bases implies that the set $\{1\}$ is contained in a basis (work out the details!). Let B be such a basis (over the rationals), and let W be the rational subspace spanned by all the remaining vectors in the basis. Then $\mathbb{R}$ is the union of the cosets $c+W$ where c runs over all rational numbers, and this is a countable union. We claim that each of these cosets has a nonempty interior. To see this, note that each coset contains exactly one rational number, while if there interior were nonempty it would contain an open interval and every open interval contains infinitely many rational numbers (*e.g.*, if $c \in (a,b)$ there is a strictly decreasing sequence of rational numbers $r_n \in (c,b)$ whose limit is c).■

4. Show that if every point in X has a neighborhood that is a a Baire space then X is a Baire space. [*Hint:* Use the open set formulation of the Baire conditions in Lemma 48.1 on page 296 of Munkres.]

SOLUTION.

According to the hint, we need to show that if $\{ V_n \}$ is a sequence of open dense subsets in X, then $(\cap_n V_n)$ is dense.

Let $x \in X$, and let W be an open neighborhood of x that is a Baire space. We claim that the open sets $V_n \cap W$ are all dense in W. Suppose that W_0 is a nonempty open subset of W; then W_0 is also open in X, and since $\cap_n V_n$ is dense in X it follows that

$$W_0 \cap (W \cap V_n) = (W_0 \cap W) \cap V_n = W_0 \cap V_n \neq \emptyset$$

for all n. Therefore $V_n \cap W$ is dense in W. Since W is a Baire space and, it follows that

$$\bigcap_n (V_n \cap W) = W \cap \left(\bigcap_n V_n \right)$$

is dense in W.

To show that $(\cap_n V_n)$ is dense in X, let U be a nonempty open subset of X, let $a \in U$, let W_a be an open neighborhood of a that is a Baire space, and let $U_0 = U \cap W$ (hence $a \in U_0$). By the previous paragraph the intersection

$$U_0 \cap \left(\bigcap_n (W \cap V_n) \right)$$

is nonempty, and since this intersection is contained in

$$U \quad \cap \quad \left(\bigcap_n V_n \right)$$

it follows that the latter is also nonempty, which implies that the original intersection $(\cap_n V_n)$ is dense.■

Problem from Munkres, § 27, *pp.* 178 – 179

6. Construct the Cantor set by taking $A_0 = [0,1]$ and defining A_n by removing

$$\bigcup_k \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

from A_{n-1}. The intersection $\cap_k A_k$ is the Cantor set C.

PRELIMINARY OBSERVATION. Each of the intervals that is removed from A_{n-1} to construct A_n is entirely contained in the former. One way of doing this is to partition $[0,1]$ into the 3^n intervals

$$\left[\frac{b}{3^n}, \frac{b+1}{3^n}\right]$$

where b is an integer between 0 and $3^n - 1$. If we write down the unique 3-adic expansion of b as a sum

$$\sum_{i=0}^{n-1} a_i \, 3^i$$

where $b_i \in \{0,1,2\}$, then A_k consists of the intervals associated to numbers b such that none of the coefficients b_i is equal to 1. Note that these intervals are pairwise disjoint; if the interval corresponding to b lies in A_k then either the interval corresponding to $b+1$ or $b-1$ is not one of the intervals that are used to construct A_k (the base 3 expansion must end with a 0 or a 2, and thus one of the adjacent numbers has a base 3 expansion ending in a 1). The inductive construction of A_n reflects the fact that for each k the middle third is removed from the closed interval

$$\left[\frac{k}{3^{n-1}}, \frac{k+1}{3^{n-1}}\right] .$$

(a) Show that C is totally disconnected; *i.e.*, the only nonempty connected subspaces are one point subsets.

SOLUTION.

By induction A_k is a union of 2^k pairwise disjoint closed intervals, each of which has length 3^{-k}; at each step one removes the middle third from each of the intervals at the previous step. Suppose that $K \subset C$ is nonempty and connected. Then for each n the set K must lie in one of the 2^n disjoint intervals of length 3^{-n} in A_n. Hence the diameter of K is $\leq 3^{-n}$ for all $n \geq 0$, and this means that the diameter is zero; *i.e.*, K consists of a single point.■

(b) Show that C is compact.

SOLUTION.

By construction C is an intersection of closed subsets o the compact space $[0,1]$, and therefore it is closed and hence compact.■

(c) Show that the endpoints of the 2^n intervals comprising A_n all lie in C.

SOLUTION.

If we know that the left and right endpoints for the intervals comprising A_n are also (respectively) left and right hand endpoints for intervals comprising A_{n+1}, then by induction it will follow that they are similar endpoints for intervals comprising A_{n+k} for all $k \geq 0$ and therefore they will all be points of C. By the descriptions given before, the left hand endpoints for A_n are all numbers of the form $b/3^n$ where b is a nonnegative integer of the form

$$\sum_{i=0}^{n-1} b_i \, 3^i$$

with $b_i = 0$ or 2 for each i, and the right hand endpoints have the form $(b+1)/3^n$ where b has the same form. If b has the indicated form then

$$\frac{b}{3^n} \quad = \quad \frac{3b}{3^{n+1}}\ , \quad \text{where} \quad 3b = \sum_{i=1}^{n} b_{i-1}\, 3^i$$

shows that $b/3^n$ is also a left hand endpoint for one of the intervals comprising A_{n+1}. Similarly, if $(b+1)/3^n$ is a right hand endpoint for an interval in A_n and b is expanded as before, then the equations

$$\frac{b+1}{3^n} \quad = \quad \frac{3b+3}{3^{n+1}}$$

and

$$(3b+3)-1 \quad = \quad 3b+2 \quad = \quad 2 + \sum_{i=1}^{n} b_{i-1}\, 3^i$$

show that $(b+1)/3^n$ is also a right hand endpoint for one of the intervals comprising A_{n+1}.∎

(d) Show that C has no isolated points.

SOLUTION.

If $x \in C$ then for each n one has a unique closed interval J_n of length 2^{-n} in A_n such that $x \in J_n$. Let $\lambda_n(x)$ denote the left hand endpoint of that interval unless x is that point, and let $\lambda_n(x)$ be the right hand endpoint in that case; we then have $\lambda_n(x) \neq x$. By construction $\left|\, \lambda_n(x) - x \,\right| < 2^{-n}$ for all n, and therefore $x = \lim_{n\to\infty} \lambda_n(x)$. On the other hand, by the preceding portion of this problem we know that $\lambda_n(x) \in C$, and therefore we have shown that x is a limit point of C, which means that x is not an isolated point.∎

(e) Show that C is uncountable.

SOLUTION.

One way to do this is by using **(d)** and the Baire Category Theorem. By construction C is a compact, hence complete, metric space. It is infinite by **(c)**, and every point is a limit point by **(d)**. Since a countable complete metric space has isolated points, it follows that C cannot be countable.

In fact, one can show that $|C| = 2^{\aleph_0}$. The first step is to note that *if $\{\, a_k \,\}$ is an infinite sequence such that $a_k \in \{0, 2\}$ for each k, then the series*

$$\sum_{k=1}^{\infty} \frac{a_k}{3^k}$$

converges and its sum lies in C.

This assertion may be verified as follows: The infinite series converges by a comparison test with the convergent series such that $a_k = 2$ for all k. Given a point as above, the partial sum

$$\sum_{k=1}^{n} \frac{a_k}{3^k}$$

is a left hand endpoint for one of the intervals comprising A_n. The original point will lie in A^n if the sum of the rest of the terms is $\leq 1/3$. But

$$\sum_{k=n+1}^{\infty} \frac{a_k}{3^k} \ \leq \ \sum_{k=n}^{\infty} \frac{2}{3^k} \quad = \quad \frac{2}{3^{n+1}} \cdot \frac{1}{(1-\frac{1}{3})} \quad = \quad \frac{1}{3}$$

so the point does lie in A_n. Since n was artitrary, this means that the sum lies in $\cap_n A_n = C$.

Returning to the original problem of determining $|C|$, we note that the set $\mathcal{A}$ of all sequences described in the assertion is in a natural 1–1 correspondence with the set of all functions from the positive integers to $\{0,1\}$. Let $\mathcal{A}_0$ be the set of all functions whose values are nonzero for infinitely many values of n, and let $\mathcal{A}_1$ be the functions that are equal to zero for all but finitely many values of n. We then have that $|\mathcal{A}_1| = \aleph_0$ and $\mathcal{A}_0$ is infinite (why?). The map sending a function in $\mathcal{A}_0$ to the associated sum of an infinite series is 1–1 (this is just a standard property of base N expansions — work out the details), and therefore we have

$$|\mathcal{A}_0| \;\leq\; |C| \;\leq\; |\mathbb{R}| \;=\; 2^{\aleph_0}$$

and

$$|\mathcal{A}_0| \;=\; |\mathcal{A}_0| + \aleph_0 \;=\; |\mathcal{A}_0| + |\mathcal{A}_1| \;=\; |\mathcal{A}| \;=\; 2^{\aleph_0}$$

which combine to imply $|C| = 2^{\aleph_0}$.■

Additional exercises

1. Let A and B be subspaces of X and Y respectively such that $A \times B$ is nowhere dense in $X \times Y$ (with respect to the product topology). Prove that either A is nowhere dense in X or B is nowhere dense in Y, and give an example to show that "or" cannot be replaced by "and."

SOLUTION.

Suppose that the conclusion is false: *i.e.*, A is not nowhere dense in X and B is not nowhere dense in Y. Then there are nonempty open sets U and V contained in the closures of A and B respectively, and thus we have

$$\emptyset \;\neq\; U \times V \;\;\subset\;\; \overline{A} \times \overline{B} \;\;=\;\; \overline{A \times B}$$

and therefore $A \times B$ is not nowhere dense in $X \times Y$.

To see that we cannot replace "or" with "and" take $X = Y = \mathbb{R}$ and let A and B be equal to $[0,1]$ and $\{0\}$ respectively. Then A is not nowhere dense in X but $A \times B$ is nowhere dense in $X \times Y$.■

2. Is there an uncountable topological space of the first category?

SOLUTION.

The space $\mathbb{R}^\infty$ has this property because it is the union of the closed nowhere dense subspaces A_n that are defined by the condition $x_i = 0$ for $i > n$.■

3. Let X be a metric space. A map $f : X \to X$ is said to be an *expanding similarity* of X if f is onto and there is a constant $C > 1$ such that

$$\mathbf{d}\,(\,f(u),\, f(v)\,) \;\;=\;\; C \cdot \mathbf{d}(u,v)$$

for all $u,\, v \in X$ (hence f is 1–1 and uniformly continuous). Prove that every expanding similarity of a complete metric space has a unique fixed point. [*Hint and comment:* Why does f have an inverse that is uniformly continuous, and why does $f(x) = x$ hold if and only if $f^{-1}(x) = x$? If $X = \mathbb{R}^n$ and a similarity is given by $f(x) = cAx + b$ where A comes from an orthogonal matrix and either $0 < c < 1$ or $c > 1$, then one can prove the existence of a unique fixed point directly using linear algebra.]

SOLUTION.

First of all, the map f is 1–1 onto; we are given that it is onto, and it is 1–1 because $u \neq v$ implies $\mathbf{d}(\, f(u),\, f(v)\,) > \mathbf{d}(u,v) > 0$. Therefore f has an inverse, at least set-theoretically, and we denote f^{-1} by T.

We claim that T satisfies the hypotheses of the Contraction Lemma. The proof of this begins with the relations

$$\mathbf{d}(\,T(u),\,T(v)\,) = \mathbf{d}(\,f^{-1}(u),\,f^{-1}(v)\,) = \frac{1}{C1}\,\mathbf{d}\left(\,f\left(f^{-1}(u)\right),\,f\left(f^{-1}(v)\right)\,\right) = \mathbf{d}(u,v)\ .$$

Since $C > 1$ it follows that $0 < 1/C < 1$ and consequently the hypotheses of the Contraction Lemma apply to our example.

Therefore T has a unique fixed point p; we claim it is also a fixed point for f. We shall follow the hint. Since T is 1–1 and onto, it follows that $x = T(\,T^{-1}(x)\,)$ and that $T(x) = x \implies x = T^{-1}(x)$; the converse is even easier to establish, for if $x = T^{-1}(x)$ the application of T yields $T(x) = x$. Since there is a unique fixed point p such that $T(p) = p$, it follows that there is a unique point, in fact the same one as before, such that $p = T^{-1}(p)$, which is equal to $f(p)$ by definition.■

THE CLASSICAL EUCLIDEAN CASE.

This has two parts. The first is that every expanding similarity of $\mathbb{R}^n$ is expressible as a so-called affine transformation $T(v) = cAv + b$ where A is given by an orthogonal matrix. The second part is to verify that each transformation of the type described has a unique fixed point. By the formula, the equation $T(x) = x$ is equivalent to the equation $x = cAx + b$, which in turn is equivalent to $(I - cA)x = b$. The assertion that T has a unique fixed point is equivalent to the assertion that this linear equation has a unique solution. The latter will happen if $I - cA$ is invertible, or equivalently if $\det(I - cA) \neq 0$, and this is equivalent to saying that c^{-1} is not an eigenvalue of A. But if A is orthogonal this means that $|Av| = |v|$ for all v and hence the only possible eigenvalues are ± 1; on the other hand, by construction we have $0 < c^{-1} < 1$ and therefore all of the desired conclusions follow. The same argument works if $0 < c < 1$, the only change being that one must substitute $c^{-1} > 1$ for $0 < c^{-1} < 1$ in the preceding sentence.■

4. Consider the sequence $\{x_n\}$ defined recursively by the formula

$$x_{n+1} \quad = \quad \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)\ .$$

If $x_0 > 0$ and $x_0^2 > a$, show that the sequence $\{\,x_n\,\}$ converges by proving that

$$\varphi(x) \quad = \quad \frac{1}{2}\left(x + \frac{a}{x}\right)$$

is a contraction mapping on $[\sqrt{a}, x_0]$.

SOLUTION.

We need to show that φ maps $[\sqrt{a}, x_0]$ into itself and that the absolute value of its derivative takes a maximum value that is less than 1.

In this example, the best starting point is the computation of the derivative, which is simply an exercise in first year calculus:

$$\varphi'(x) \quad = \quad \frac{1}{2}\left(1 - \frac{a}{x^2}\right)$$

This expression is an increasing function of x over the set $[\sqrt{a}, +\infty)$; its value at $\sqrt{a}$ is 0 and the limit at $+\infty$ is $1/2$. In particular, the absolute value of the derivative on $[\sqrt{a}, x_0]$ is less than $1/2$, and by the Mean Value Theorem the latter in turn implies that φ maps the interval in question to

$$\left[\sqrt{a}, \frac{\sqrt{a}+x_0}{2}\right]$$

which is contained in $[\sqrt{a}, x_0]$.■

III.4 : Connected spaces

Problems from Munkres, § 23, *p.* 152

3. Let A be a connected subspace of X, and let $\{ A_\alpha \}$ be a collection of connected subspaces of X whose union is B. Show that if $A \cap A_\alpha \neq \emptyset$ for each α then $A \cup B$ is connected.

SOLUTION.

Suppose that C is a nonempty open and closed subset of $Y = A \cup B$. Then by connectedness either $A \subset C$ or $A \cap C = \emptyset$; without loss of generality we may assume the first holds, for the argument in the second case will follow by interchanging the roles of C and $Y - C$. We need to prove that $C = Y$.

Since each A_α for each α we either have $A_\alpha \subset C$ or $A_\alpha \cap C = \emptyset$. In each case the latter cannot hold because $A \cap A_\alpha$ is a nonempty subset of C, and therefore A_α is contained in C for all α. Therefore $C = Y$ and hence Y is connected.■

4. Show that if X is an infinite set then it is connected in the finite complement topology.

SOLUTION.

Suppose that we can write $X = C \cup D$ where C and D are disjoint nonempty open and closed subsets. Since C is open it follows that either D is finite or all of X; the latter cannot happen because $X - D = C$ is nonempty. On the other hand, since C is closed it follows that either C is finite or $C = X$. Once again, the latter cannot happen because $X - C = D$ is nonempty. Thus X is a union of two finite sets and must be finite, which contradicts our assumption that X is infinite. This forces X to be connected.■

5. A space is *totally disconnected* if its connected components are all one point subsets. Show that a discrete space is totally disconnected. Does the converse hold?

SOLUTION.

Suppose that C is a maximal connected subset; then C also has the discrete topology, and the only discrete spaces that are connected are those with at most one point.

There are many examples of totally disconnected spaces that are not discrete. The set of rational numbers and all of its subspaces are fundamental examples; here is the proof: Let x and y be distinct rational numbers with $x < y$. Then there is an irrational number r between them, and the identity

$$\mathbb{Q} = \Big(\mathbb{Q} \cap (-\infty, r)\Big) \quad \cup \quad \Big(\mathbb{Q} \cap (r, +\infty, r)\Big)$$

gives a separation of $\mathbb{Q}$, thus showing that x and y lie in different connected components of $\mathbb{Q}$. But x and y were arbitrary so this means that no pair of distinct points can lie in the same connected component of Q. The argument for subsets of $\mathbb{Q}$ proceeds similarly.■

9. Let A and B be proper subsets of the connected spaces X and Y. Prove that $X \times Y - A \times B$ is connected.

SOLUTION.

A good way to approach this problem is to begin by drawing a picture in which $X \times Y$ is a square and $A \times B$ is a smaller concentric square. It might be helpful to work with this picture while reading the argument given here.

Let $x_0 \in X - A$ and $y_0 \in Y - B$, and let C be the connected component of (x_0, y_0) in $X \times Y - A \times B$. We need to show that C is the entire space, and in order to do this it is enough to show that given any other point (x, y) in the space there is a connected subset of $X \times Y - A \times B$ containing it and (x_0, y_0). There are three cases depending upon whether or not $x \in A$ or $y \in B$ (there are three options rather than four because we know that both cannot be true).

If $x \notin A$ and $y \notin B$ then the sets $X \times \{y_0\}$ and $\{x\} \times Y$ are connected subsets such that (x_0, y_0) and (x, y_0) lie in the first subset while (x, y_0) and (x, y) lie in the second. Therefore there is a connected subset containing (x, y) and (x_0, y_0) by Exercise 3.— Now suppose that $x \in A$ but $y \notin B$. Then the two points in question are both contained in the connected subset $X \times \{y\} \cup times\{x_0\} \times Y$. Finally, if $x \notin A$ but $y \in B$, then the two points in question are both contained in the connected subset $X \times \{y_0\} \cup \{x\} \times Y$. Therefore the set $X \times Y - A \times B$ is connected.■

12. [*Assuming Y is a closed subset of X.*] Let Y be a nonempty [*closed*] subset of X, and assume that A and B form a separation of $X - Y$. Prove that $Y \cup A$ and $Y \cup B$ are connected. [Munkres does not explicitly assume $Y \neq \emptyset$, but without this assumption the conclusion is false.]

SOLUTION.

Since $X - Y$ is open in X and A and B are disjoint open subsets of $X - Y$, it follows that A and B are open in X. The latter in turn implies that $X \cup A$ and $Y \cup B$ are both closed in X.

We shall only give the argument for $X \cup A$; the proof for $X \cup B$ is the similar, the only change being that the roles of A and B are interchanged. Once again, it might be helpful to draw a picture.

Suppose that C is a nonempty proper subset of $Y \cup A$ that is both open and closed, and let $D = (Y \cup A) - C$. One of the subsets C, D must contain some point of Y, and without loss of generality we may assume it is C. Since Y is connected it follows that all of Y must be contained in C. Suppose that $D \neq \emptyset$. Since D is closed in $Y \cup A$ and the latter is closed in X, it follows that D is closed in X. On the other hand, since D is open in $Y \cup A$ and disjoint from Y it follows that D is open in A, which is the complement of Y in $Y \cup A$. But A is open in X and therefore D is open in X. By connectedness we must have $D = X$, contradicting our previous observation that $D \cap Y = \emptyset$. This forces the conclusion that D must be empty and hence $Y \cup A$ is connected.■

Problems from Munkres, § 24, *pp.* 157 – 159

1. **(b)** Suppose that there exist embeddings $f : X \to Y$ and $g : Y \to X$. Show by means of an example that X and Y need not be homeomorphic.

SOLUTION.

Take $X = (0, 1) \cup (2, 3)$ and $Y = (0, 3)$, let f be inclusion, and let g be multiplication by $1/3$. There are many other examples. In the spirit of part **(a)** of this exercise, one can also take $X = (0, 1)$, $Y = (0, 1]$, f = inclusion and g = multiplication by $1/2$, and similarly one can take $X = (0, 1]$, $Y = [0, 1]$, f = inclusion and $g(t) = (t + 1)/2$.■

FOOTNOTE.

Notwithstanding the sort of examples described in the exercises, a result of S. Banach provides a "resolution" of X and Y if there are continuous embeddings $f : X \to Y$ and $g : Y \to X$; namely, there are decompositions $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ such that $X_1 \cap X_2 = Y_1 \cap Y_1 = \emptyset$ and there are homeomorphisms $X_1 \to X_2$ and $Y_1 \to Y_2$. The reference is S. Banach, *Un théorème sur les transformations biunivoques*, Fundamenta Mathematicæ **6** (1924), 236–239.

Additional exercises

1. Prove that a topological space X is connected if and only if every open covering $\mathcal{U} = \{U_\alpha\}$ has the following property: For each pair of sets U, V in $\mathcal{U}$ there is a sequence of open sets $\{U_0, U_1, \cdots U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i.

SOLUTION.

Define a binary relation $\sim$ on X such that $u \sim v$ if and only if there are open subsets U and V in $\mathcal{U}$ such that $u \in U$, $v \in V$, and there is a sequence of open sets $\{U_0, U_1, \cdots U_n\}$ in $\mathcal{U}$ such that $U = U_0$, $V = U_n$, and $U_i \cap U_{i+1} \neq \emptyset$ for all i. This is an equivalence relation (verify this in detail!). Since every point lies in an open subset that belongs to $\mathcal{U}$ it follows that the equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence classs is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

2. Let X be a connected space, and let $\mathcal{R}$ be an equivalence relation that is locally constant (for each point x all points in some neighborhood of x lie in the $\mathcal{R}$-equivalence class of x). Prove that $\mathcal{R}$ has exactly one equivalence class.

SOLUTION.

This is similar to some arguments in the course notes and to the preceding exercise. The hypothesis that the relation is locally constant implies that equivalence classes are open. Therefore the union of all but one equivalence class is also open, and hence a single equivalence classs is also closed in the space. If X is connected, this can only happen if there is exactly one equivalence class.■

3. Prove that an open subset in $\mathbb{R}^n$ can have at most countably many components. Give an example to show this is not necessarily true for closed sets.

SOLUTION.

The important point is that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$; given a point $x \in \mathbb{R}^n$ with coordinates x_i and $\varepsilon > 0$ we can choose rational numbers q_i such that $|q_i - x_i| < \varepsilon/n$; if y has coordinates q_i then $|x - y| < \varepsilon$ follows immediately. Since $\mathbb{R}^n$ is locally connected, the components of open sets are open, and therefore we conclude that every component of a nonempty open set $U \subset \mathbb{R}^n$ contains some point of $\mathbb{Q}^n$. Picking one such point for each component we obtain a 1–1 map from the set of components into the countable set $\mathbb{Q}^n$.

The Cantor set is an example of a closed subset of $\mathbb{R}$ for which the components are the one point subsets and there are uncountably many points.■

III.5 : Variants of connectedness

Problem from Munkres, § 24, pp. 157 – 159

8. **(b)** If $A \subset X$ and A is path connected, is $\overline{A}$ path connected?

SOLUTION.

Not necessarily. Consider the graph of $\sin(1/x)$ for $x > 0$. It closure is obtained by adding the points in $\{0\} \times [-1, 1]$ and we have shown that the space consisting of the graph and this closed segment is not path connected.■

Problem from Munkres, § 25, pp. 162 – 163

10. [*Only* **(a)**, **(b)** *and examples A and B from* **(c)**.] Let X be a space, and define a binary relation by $x \sim y$ if there is no separation $X = A \cup B$ into disjoint open subsets such that $x \in A$ and $y \in B$.

(a) Show that this is an equivalence relation; the equivalence classes are called *quasicomponents.*

SOLUTION.

The proofs that $\sim$ is reflexive and symmetric are very elementary and left to the reader. Regarding transitivity, suppose that $x \sim y$ and $y \sim z$ and that we are given a separation of X as $A \cup B$. Without loss of generality we may assume that $x \in A$ (otherwise interchange the roles of A and B in the argument). Since $x \sim y$ it follows that $y \in A$, and the latter combines with $y \sim z$ to show that $z \in A$. Therefore if we are given a separation of X as above, then x and z lie in the same piece, and since this happens for all separations it follows that $x \sim z$.■

(b) Show that each [connected] component of X is contained in a quasicomponent and that the components and quasicomponents are the same if X is locally connected.

SOLUTION.

Let C be a connected component of X. Then $A \cap C$ is an open and closed subset of C and therefore it is either empty or all of C. In the first case $C \subset B$ and in the second case $C \cap B$. In either case it follows that all points of C lie in the same equivalence class. – If X is locally connected then each connected component is both closed and open. Therefore, if x and y lie in different components, say C_x and C_y, then $X = C_x \cup X - C_x$ defines a separation such that x lies in the first subset and y lies in the second, so that $x \sim y$ is false if x and y do not lie in the same connected component. Combining this with the previous part of the exercise, if X is locally connected then $x \sim y$ if and only if x and y lie in the same connected component.■

(c) Let K denote the set of all reciprocals of positive integers, and let $-K$ denote the set $(-1)\cdot K$ = all negatives of points in K. Determine the components, path components and quasicomponents of the following subspaces of $\mathbb{R}^2$:

$$A = (K \times [0,1]) \cup \{(0,0)\} \cup \{(0,1)\}$$

$$B = A \cup ([0,1] \times \{0\})$$

[*Note:* Example C in this part of the problem was not assigned; some comments appear below.]

SOLUTION.

Trying to prove this without any pictures would probably be difficult at best and hopelessly impossible at worst.

Case A**.** We claim that the connected components are the closed segments $\{1/n\} \times [0,1]$ and the one point subsets $\{(0,0)\}$ and $\{(0,1)\}$. — Each segment is connected and compact (hence closed), and we claim each is also an open subset of A. This follows because the segment $\{1/n\}\times[0,1]$ is the intersection of A with the open subset

$$\left(\frac{1}{2}\left[\frac{1}{n}+\frac{1}{n+1}\right] \ , \ \frac{1}{2}\left[\frac{1}{n}+\frac{1}{n-1}\right]\right) \times \mathbb{R} \ .$$

Since the segment $\{1/n\} \times [0,1]$ is open, closed and connected, it follows that it must be both a component and a quasicomponent. It is also an arc component because it is arcwise connected. This leaves us with the two points $(0,0)$ and $(0,1)$. They cannot belong to the same component because they do not form a connected set. Therefore each belongs to a separate component and also to a separate path component. The only remaining question iw whether or not they determine the same quasicomponent. To show that they do lie in the same quasicomponent, it suffices to check that if $A = U \cup V$ is a separation of X into disjoint open subsets then both points lie in the same open set. Without loss of generality we may as well assume that the origin lies in U. It then follows that for all n sufficiently large the points $(1/n, 0)$ all lie in U, and the latter implies that all of the connected segments $\{1/n\} \times [0,1]$ is also lie in U for n sufficiently large. Since $(0,1)$ is a limit point of the union of all these segments (how?), it follows that $(0,1)$ also lies in U. This implies that $(0,1)$ lies in the same quasicomponent of A as $(0,0)$.∎

Case B**.** This set turns out to be connected but not arcwise connected. We claim that the path components are given by $\{(0,1)\}$ and its complement. Here is the proof that the complement is path connected: Let P be the path component containing all points of $[0,1] \times \{0\}$. Since the latter has a nontrivial intersection with each vertical closed segment $\{1/n\} \times [0,1]$ it follows that all of these segments are also contained in P, and hence P consists of all points of A except perhaps $(0,1)$. Since $(0,1)$ is a limit point of P (as before) it follows that B is connected and thus there is only one component and one quasicomponent. We claim that there are two path components. Suppose the extra point $(0,1)$ also lies in P. Then, for example, there will be a continuous curve $\alpha : [0,1] \to A$ joining $(0,1)$ to $(1/2,1)$. Let t_0 be the maximum point in the subset of $[0,1]$ where the first coordinate is zero. Since the first coordinate of $\alpha(a)$ is 1, we must have $t_0 < 1$. Since $A \cap \{0\} \times \mathbb{R}$ is equal to $\{(0,0)\} \cup \{(0,1)\}$, it follows that α is constant on $[0, t_0]$, and by continuity there is a $\delta > 0$ such that $|t - t_0| < \delta$ implies that the *second* coordinate of $\alpha(t)$ is greater than $1/2$. If $t \in (t_0, t_0 + (\delta/2))$, then the first coordinate of $\alpha(t)$ is positive and the second is greater than $1/2$. In fact we can choose some t_1 in the open interval such that the first coordinate is irrational. But there are no points in $B \cap (\frac{1}{2}, +\infty)$ whose first coordinates are irrational, so we have a contradiction. The latter arises because we assumed the existence of a continuous curve joining $(0,1)$ to another point in B. Therefore no such curve can exist and $(0,1)$ does not belong to the path component P. Hence B has two path components, and one of them contains only one point.∎

Case C**.** This was not assigned, but we note that this space is connected and each of the (infinitely many) closed segments given in the definition is a separate path component.∎

Additional exercises

1. Prove that a compact locally connected space has only finitely many components.

SOLUTION.

In a locally connected space the connected components are open (and pairwise disjoint). These sets form an open covering and by compactness there is a finite subcovering. Since no proper subcollection of the set of components is an open covering, this implies that the set of components must be finite.■

2. Give an example to show that if X and Y are locally connected metric spaces and $f : X \to Y$ is continuous then $f[X]$ is not necessarily locally connected.

SOLUTION.

Let $Y = \mathbb{R}^2$ and let $X \subset \mathbb{R}^2$ be the union of the horizontal half-line $(0, \infty) \times \{0\}$ and the vertical closed segment $\{-1\} \times [-1, 1]$. These subsets of X are closed in X and pairwise disjoint. Let $f : X \to \mathbb{R}^2$ be the continuous map defined on $(0, \infty) \times \{0\}$ by the formula $f(t, 0) = (t, \sin(1/t))$ and on $\{-1\} \times [-1, 1]$ by the formula $f(-1, s) = (0, s)$. The image $f[X]$ is then the example of a non-locally connected space that is described in the course notes.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 4

Fall 2008

IV. Function spaces

IV.1 : General properties

(Munkres, §§ 45–47)

Additional exercises

1. Suppose that X and Y are metric spaces such that X is compact. Let Y^X denote the cartesian product of the spaces $Y \times \{x\} \cong Y$ with the product topology, and for each $x \in X$ let $p_x : Y^X \to Y$ denote projection onto the factor corresponding to x. Let $q : \mathbf{C}(X,Y) \to Y^X$ be the map such that for each x the composite $p_x \circ q$ sends f to $f(x)$. Prove that q is a continuous 1–1 mapping.

SOLUTION.

The mapping q is 1–1 because $q(f) = q(g)$ implies that for all x we have $f(x) = p_x \circ q(f) = p_x \circ q(g) = g(x)$, which means that $f = g$.

To prove continuity, we need to show that the inverse images of subbasic open sets in Y^X are open in $\mathbf{C}(X,Y)$. The standard subbasic open subsets have the form $\mathcal{W}(\{x\}, U) = p_x^{-1}(U)$ where $x \in X$ and U is open in Y. In fact, there is a smaller subbasis consisting of all such sets $\mathcal{W}(\{x\}, U)$ such that $U = N_\varepsilon(y)$ for some $y \in Y$ and $\varepsilon > 0$. Suppose that f is a continuous function such that $q(f)$ lies in $\mathcal{W}(\{x\}, U)$. By definition the later condition means that $f(x) \in U$. The latter in turn implies that $\delta = \varepsilon - \mathbf{d}(\,f(x), y) > 0$, and if $\mathbf{d}(f,g) < \delta$ then the Triangle Inequality implies that $\mathbf{d}(\,g(x), y) < \varepsilon$, which in turn means that $g(x) \in U$. Therefore q is continuous at f, and since f is arbitrary this shows q is a continuous mapping.■

2. Suppose that X, Y and Z are metric spaces such that X is compact, and let p_Y, p_Z denote projections from $Y \times Z$ to Y and Z respectively. Assume that one takes the $\mathbf{d}_\infty$ or maximum metric on the product (*i.e.*, the distance between (x_1, y_1) and (x_2, y_2) is the larger of $\mathbf{d}(x_1, y_1)$ and $\mathbf{d}(x_2, y_2)$). Prove that the map from $\mathbf{C}(X, Y \times Z)$ to $\mathbf{C}(X,Y) \times \mathbf{C}(X,Z)$ sending f to $(p_y \circ f, p_Z \circ f)$ is an isometry (where the codomain also has the corresponding $\mathbf{d}_\infty$ metric).

SOLUTION.

It suffices to show that the map in question is onto and distance-preserving. The map is onto because if u and v are continuous functions into Y and Z respectively, then we can retrieve f by the formula $f(x) = \big(\,u(x),\, v(x)\,\big)$. Suppose now that f and g are continuous functions from X to $Y \times Z$. Then the distance from f to g is the maximum of $\mathbf{d}\big(\,f(x),\, g(x)\big)$. The latter is less than or equal to the greater of $\mathbf{d}\big(\,p_Y f(s),\, p_Y g(s)\,\big)$ and $\mathbf{d}\big(\,p_Y f(t),\, p_Y g(t)\,\big)$. Thus if

$$\Phi : \mathbf{C}(X, Y \times Z) \to \mathbf{C}(X,Y) \times \mathbf{C}(X,Z)$$

then the distance from $\Phi(f)$ to $\Phi(g)$ is greater than or equal to the distance from f to g. This means that the map Φ^{-1} is uniformly continuous. Conversely, we claim that the distance from f to g is greater than or equal to the distance between $\Phi(f)$ and $\Phi(g)$. The latter is equal to the larger of the maximum values of $\mathbf{d}\big(p_Y \circ f(s),\, p_Y \circ g(s)\big)$ and $\mathbf{d}\big(p_Z \circ f(t),\, p_Z \circ g(t)\big)$. If $w \in X$ is where $\mathbf{d}(f, g)$ takes its maximum, it follows that

$$\mathbf{d}\big(\, f(w),\, g(t)\,\big) \quad =$$

$$\max\Big\{\, \mathbf{d}\big(p_Y \circ f(w),\, p_Y \circ g(w)\big),\ \mathbf{d}\big(p_Z \circ f(w),\, p_Z \circ g(w)\big) \,\Big\}$$

which is less than or equal to the distance between $\Phi(f)$ and $\Phi(g)$ as described above.$\blacksquare$

3. Suppose that X and Y are metric spaces such that X is compact, and assume further that X is a union of two disjoint open and closed subsets A and B. Prove that the map from $\mathbf{C}(X, Y)$ to $\mathbf{C}(A, Y) \times \mathbf{C}(B, Y)$ sending f to $(f|A, f|B)$ is also an isometry, where as before the product has the $\mathbf{d}_\infty$ metric.

SOLUTION.

The distance between f and g is the maximum value of the distance between $f(x)$ and $g(x)$ as x runs through the elements of x, which is the greater of the maximum distances between $f(x)$ and $g(x)$ as x runs through the elements of C, where C runs through the set $\{A, B\}$. But the second expression is equal to the larger of the distances between $\mathbf{d}(f|A, g|A)$ and $\mathbf{d}(f|B, g|B)$. Therefore the map described in the problem is distance preserving. As in the previous exercise, to complete the proof it will suffice to verify that the map is onto. The surjectivity is equivalent to saying that a function is continuous if its restrictions to the closed subsets A and B are continuous. But we know the latter is true.$\blacksquare$

4. Suppose that X, Y, Z, W are metric spaces such that X and Z are compact. Let $P :$ $\mathbf{C}(X, Y) \times \mathbf{C}(Z, W) \to \mathbf{C}(X \times Z, Y \times W)$ be the map sending (f, g) to the product map $f \times g$ (recall that $f \times g(x, y) = (f(x), g(y))$). Prove that P is continuous, where again one uses the associated $\mathbf{d}_\infty$ metrics on all products.

SOLUTION.

Let $\varepsilon > 0$ be given. We claim that the distance between $f \times g$ and $f' \times g'$ is less than ε if the distance between f and f' is less than ε and the distance between g and g' is less than ε. Choose $u_0 \in X$ and $v_0 \in Z$ so that

$$\mathbf{d}\big(\, f' \times g'(u_0, v_0),\ f \times g(u_0, v_0),\,\big)$$

is maximal and hence equal to the distance between $f \times g$ and $f' \times g'$. The displayed quantity is equal to the greater of $\mathbf{d}\big(\, f'(u_0),\, f(u_0)\big)$ and $\mathbf{d}\big(\, g'(v_0),\, g(v_0)\big)$. These quantities in turn are less than or equal to $\mathbf{d}(f', f)$ and $\mathbf{d}(g', g)$ respectively. Therefore if both of the latter are less than ε it follows that the distance between $f' \times g'$ and $f \times g$ is less than ε.$\blacksquare$

5. Suppose that X and Y are metric spaces and $f : X \to Y$ is a homeomorphism.

(i) If A is a compact metric space, show that the map $V(f)$ defines a homeomorphism from $\mathbf{C}(A, X)$ to $\mathbf{C}(A, Y)$. [*Hint:* Consider $V(h)$, where $h = f^{-1}$.]

SOLUTION.

Follow the hint. We then have $V(h) \circ V(f) = V(h \circ f) = V(\mathrm{id})$, which is the identity. Likewise, we also have $V(f) \circ V(h) = V(f \circ h) = V(\mathrm{id})$, which is the identity.$\blacksquare$

(ii) If X and Y are compact and B is a metric space, show that the map $U(f)$ defines a homeomorphism from $\mathbf{C}(Y,B)$ to $\mathbf{C}(X,B)$. [*Hint:* Consider $U(h)$, where $h = f^{-1}$.]

SOLUTION.

Again follow the hint. We then have $U(f) \circ U(h) = U(h \circ f) = V(\mathrm{id})$, which is the identity. Likewise, we also have $U(h) \circ V(f) = V(f \circ h) = V(\mathrm{id})$, which is the identity.■

(iii) Suppose that A and A' are homeomorphic compact metric spaces and B and B' are homeomorphic metric spaces. Prove that $\mathbf{C}(A,B)$ is homeomorphic to $\mathbf{C}(A',B')$.

SOLUTION.

Let $f : A \to A'$ and $g : B \to B'$ be the homeomorphisms. Let $U(f^{-1}) \circ V(g) = V(g) \circ U(f^{-1})$ — equality holds by associativity of composition. By the first two parts of the exercise this map is the required homeomorphism from $\mathbf{C}(A,B)$ to $\mathbf{C}(A',B')$.■

6. Given a metric space X and $a, b \in X$, let $\mathbf{P}(X; a, b)$ be the space of all continuous functions or curves γ from $[0,1]$ to X such that $\gamma(0) = a$ and $\gamma(1) = b$. Given $a, b, c \in X$ define a concatenation map

$$\alpha : \mathbf{P}(X; a, b) \times \mathbf{P}(X; b, c) \longrightarrow \mathbf{P}(X; a, c)$$

such that $\alpha(\gamma, \gamma')(t) = \gamma(2t)$ if $t \leq \frac{1}{2}$ and $\gamma'(2t-1)$ if $t \geq \frac{1}{2}$. Informally, this is a reparametrization of the curve formed by first going from a to b by γ, and then going from b to c by γ'. **Prove** that this concatenation map is uniformly continuous (again take the $\mathbf{d}_\infty$ metric on the product).

SOLUTION.

It will be convenient to denote $\alpha(\gamma, \gamma')$ generically by $\gamma + \gamma'$. The construction then implies that the distance between $\xi + \xi'$ and $\eta + \eta'$ is the larger $\mathbf{d}(\xi, \eta)$ and $\mathbf{d}(\xi', \eta')$. Therefore the concatenation map is distance preserving.

7. Let X and Y be topological spaces, take the compact-open topology on $\mathbf{C}(X,Y)$, and let $k : Y \to \mathbf{C}(X,Y)$ be the map sending a point $y \in Y$ to the constant function $k(y)$ whose value at every point is equal to y. Prove that k maps Y homeomorphically to $k(Y)$. [*Hints:* First verify that k is 1–1. If $\mathcal{W}(K,U)$ is a basic open set for the compact-open topology with $K \subset X$ compact and $U \subset Y$ open, prove that $k^{-1}[\,\mathcal{W}(K,U)\,] = U$ and $k[U] = \mathcal{W}(K,U) \cap \mathrm{Image}(k)$. Why do the latter identities imply the conclusion of the exercise?]

SOLUTION.

Follow the hint. If $y \neq y'$, then the values of $k(y)$ at every point is X, and hence it is not equal to y', the value of $k(y')$ at every point of X. Therefore k is 1–1.

Next, we shall verify the set-theoretic identities described above. If $y \in U$ then since $k(y)$ is the function whose value is y at every point we clearly have $k(y) \in \mathcal{W}(K,U)$, and hence $k(U) \subset \mathcal{W}(K,U) \cap \mathrm{Image}(k)$. Conversely, any constant function in the image of the latter is equal to $k(y)$ for some $y \in U$. The identity $k^{-1}(\,\mathcal{W}(K,U)\,) = U$ follows similarly.■

IV.2 : Adjoint equivalences

(Munkres, §§ 45–46)

Additional exercises

1. State and prove an analog of the main result of this section for in which X, Y, Z are untopologized sets (with no cardinality restrictions!) and spaces of continuous functions are replaced by sets of set-theoretic functions.

SOLUTION.

The map $A : \mathbf{F}(X \times Y, Z) \to \mathbf{F}(X,\ \mathbf{F}(Y, Z)\,)$ sends $h : X \times Y \to Z$ to the function $h^\flat$ such that $[h^\flat(x)](y) = h(x, y)$. The argument proving the adjoint formula for spaces of continuous functions modifies easily to cover these examples, and in fact in this case the proof is a bit easier because it is not necessary to consider metrics or topologies.

2. Suppose that X is a compact metric space and Y is a convex subset of $\mathbb{R}^n$. Prove that $\mathbf{C}(X, Y)$ is arcwise connected. [*Hint:* Let $f : X \to Y$ be a continuous function. Pick some $y_0 \in Y$ and consider the map $H : X \times [0, 1] \to Y$ sending (x, t) to the point $(1 - t)f(x) + ty_0$ on the line segment joining $f(x)$ to y_0.]

SOLUTION.

Let $y_0 \in Y$, and let $L : Y \times [0, 1] \to Y$ be the map sending (y, t) to the point $(1 - t)y + ty_0$ on the line segment joining y to y_0. If we set $H(x, t) = L(\,f(x), t)$, then H satisfies the conditions in the hint and defines a continuous map in $\mathbf{C}(X, Y)$ joining f to the constant function whose values is always y_0. Thus for each f we know that f and the constant function with value y_0 lie in the same arc component of $\mathbf{C}(X, Y)$. Therefore there must be only one arc component.$\blacksquare$

3. Suppose that X, Y, Z are metric spaces such that X and Y are compact. Prove that there is a 1–1 correspondence between continuous functions from X to $\mathbf{C}(Y, Z)$ and continuous functions from Y to $\mathbf{C}(X, Z)$. As usual, assume the function spaces have the topologies determined by the uniform metric.

SOLUTION.

By the adjoint formula there are homeomorphisms

$$\mathbf{C}\big(X,\ \mathbf{C}(Y, Z)\big) \quad \cong \quad \mathbf{C}(X \times Y, Z) \quad \cong \quad \mathbf{C}(Y \times X, Z) \quad \cong \quad \mathbf{C}\big(Y,\ \mathbf{C}(X, Z)\big)$$

and this yields the desired 1–1 correspondence of sets.$\blacksquare$

V. Constructions on spaces

V.1 : Quotient spaces

Problem from Munkres, § 22*, pp.* 144 − 145

4. **(a)** Define an equivalence relation on the Euclidean coordinate plane by defining $(x_0, y_0) \sim (x_1, y_1)$ if and only if $x_0 + y_0^2 = x_1 + y_1^2$. It is homeomorphic to a familiar space. What is it? [*Hint:* Set $g(x, y) = x + y^2$.]

SOLUTION.

The hint describes a well-defined continuous map from the quotient space W to the real numbers. The equivalence classes are simply the curves $g(x, y) = C$ for various values of C, and they are parabolas that open to the left and whose axes of symmetry are the x-axis. It follows that there is a 1–1 onto continuous map from W to $\mathbb{R}$. How do we show it has a continuous inverse? The trick is to find a continuous map in the other direction. Specifically, this can be done by composing the inclusion of $\mathbb{R}$ in $\mathbb{R}^2$ as the x-axis with the quotient projection from $\mathbb{R}^2$ to W. This gives the set-theoretic inverse to $\mathbb{R}^2 \to W$ and by construction it is continuous. Therefore the quotient space is homeomorphic to $\mathbb{R}$ with the usual topology.■

(b) Answer the same question for the equivalence relation $(x_0, y_0) \sim (x_1, y_1)$ if and only if $x_0^2 + y_0^2 = x_1^2 + y_1^2$.

SOLUTION.

Here we define $g(x, y) = x^2 + y^2$ and the equivalence classes are the circles $g(x, y) = C$ for $C > 0$ along with the origin. In this case we have a continuous 1–1 onto map from the quotient space V to the nonnegative real numbers, which we denote by $[0, \infty)$ as usual. To verify that this map is a homeomorphism, consider the map from $[0, \infty)$ to V given by composing the standard inclusion of the former as part of the x-axis with the quotient map $\mathbb{R}^2 \to V$. This is a set-theoretic inverse to the map from V to $[0, \infty)$ and by construction it is continuous.■

Additional exercises

0. Suppose that X is a space with the discrete topology and $\mathcal{R}$ is an equivalence relation on X. Prove that the quotient topology on $X/\mathcal{R}$ is discrete.

SOLUTION.

We claim that every subset of $X/\mathcal{R}$ is both open and closed. But a subset of the quotient is open and closed if and only if the inverse image has these properties, and every subset of a discrete space has these properties.■

1. If A is a subspace of X, a continuous map $r : X \to A$ is called a *retraction* if the restriction of r to A is the identity. Show that a retraction is a quotient map.

SOLUTION.

We need to show that $U \subset A$ is open if and only if $r^{-1}[U]$ is open in X. The ($\Longrightarrow$) implication is immediate from the continuity of r. To prove the other direction, note that $r \circ i = \mathrm{id}_A$ implies that

$$U \quad = \quad i^{-1}\left[r^{-1}[U]\right] \quad = \quad A \cap r^{-1}[U]$$

and thus if the inverse image of U is open in X then U must be open in A.■

2. Let $\mathcal{R}$ be an equivalence relation on a space X, and assume that $A \subset X$ contains points from every equivalence class of $\mathcal{R}$. Let $\mathcal{R}_0$ be the induced equivalence relation on A, and let

$$j : A/\mathcal{R}_0 \longrightarrow X/\mathcal{R}$$

be the associated $1-1$ correspondence of equivalence classes. Prove that j is a homeomorphism if there is a retraction $r : X \to A$ sending $\mathcal{R}$-equivalent points of X to $\mathcal{R}_0$-equivalent points of A. such that each set $r^{-1}[\,\{a\}\,]$ is contained in an $\mathcal{R}$-equivalence class.

SOLUTION.

Let p_X and p_A denote the quotient space projections for X and A respectively. By construction, j is the unique function such that $j \circ p_A = p_X|A$ and therefore j is continuous. We shall define an explicit continuous inverse $k : X/\mathcal{R} \to A/\mathcal{R}_0$. To define the latter, consider the continuous map $p_A \circ r : X \to A/\mathcal{R}_0$. If $y\mathcal{R}z$ holds then $r(y)\mathcal{R}_0\, r(z)$, and therefore the images of y and z in $A/\mathcal{R}_0$ are equal. Therefore there is a unique continuous map k of quotient spaces such that $k \circ p_X = p_A \circ r$. This map is a set-theoretic inverse to j and therefore j is a homeomorphism.■

3. **(a)** Let 0 denote the origin in $\mathbb{R}^3$. In $\mathbb{R}^3 - \{0\}$ define $x\mathcal{R}y$ if y is a nonzero multiple of x (geometrically, if x and y lie on a line through the origin). Show that $\mathcal{R}$ is an equivalence relation; the quotient space is called the *real projective plane* and denoted by $\mathbb{R}\mathbf{P}^2$.

SOLUTION.

The relation is reflexive because $x = 1 \cdot x$, and it is reflexive because $y = \alpha x$ for some $\alpha \neq 0$ implies $x = \alpha^{-1} y$. The relation is transitive because $y = \alpha x$ for $\alpha \neq 0$ and $z = \beta y$ for $y \neq 0$ implies $z = \beta\alpha x$, and $\beta\alpha \neq 0$ because the product of nonzero real numbers is nonzero.■

(b) Using the previous exercise show that $\mathbb{RP}^2$ can also be viewed as the quotient of S^2 modulo the equivalence relation $x \sim y \iff y = \pm\, x$. In particular, this shows that $\mathbb{RP}^2$ is compact. [*Hint:* Let r be the radial compression map that sends v to $|v|^{-1}v$.]

SOLUTION.

Use the hint to define r; we may apply the preceding exercise if we can show that for each $a \in S^2$ the set $r^{-1}(\{a\})$ is contained in an $\mathcal{R}$-equivalence class. By construction $r(v) = |v|^{-1}v$, so $r(x) = a$ if and only if x is a positive multiple of a (if $x = \rho\, a$ then $|x| = \rho$ and $r(x) = a$, while if $a = r(x)$ then by definition a and x are positive multiples of each other). Therefore if $x\mathcal{R}y$ then $r(x) = \pm\, r(y)$, so that $r(x)\mathcal{R}_0\, r(y)$ and the map

$$S^2/[x \equiv \pm x] \quad \longrightarrow \quad \mathbb{RP}^2$$

is a homeomorphism.■

4. In $D^2 = \{x \in \mathbb{R}^2 \mid |x| \leq 1\}$, consider the equivalence relation generated by the condition $x\mathcal{R}'y$ if $|x| = |y| = 1$ and $y = -x$. Show that this quotient space is homeomorphic to $\mathbb{RP}^2$.

[*Hints:* Use the description of $\mathbb{RP}^2$ as a quotient space of S^2 from the previous exercise, and let $h : D^2 \to S^2$ be defined by

$$h(x, y) \quad = \quad (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces. Why is $\overline{h}$ a $1-1$ and onto mapping? Finally, prove that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.]

SOLUTION.

Needless to say we shall follow the hints in a step by step manner.

Let $h : D^2 \to S^2$ be defined by

$$h(x,y) = (x, y, \sqrt{1 - x^2 - y^2}\,).$$

Verify that h preserves equivalence classes and therefore induces a continuous map $\overline{h}$ on quotient spaces.

To show that $\overline{h}$ is well-defined it is only necessary to show that its values on the $\mathcal{R}'$-equivalence classes with two elements are the same for both representatives. If $\pi : S^2 \to \mathbb{RP}^2$ is the quotient projection, this means that we need $\pi \circ h(u) = \pi \circ h(v)$ if $|u| = |v| = 1$ and $u = -v$. This is immediate from the definition of the equivalence relation on S^2 and the fact that $h(w) = w$ if $|w| = 1$.

Why is $\overline{h}$ a $1-1$ and onto mapping?

By construction h maps the equivalence classes of points on the unit circle onto the points of S^2 with $z = 0$ in a 1–1 onto fashion. On the other hand, if u and v are distinct points that are not on the unit circle, then $h(u)$ cannot be equal to $\pm\, h(v)$. The inequality $h(u) \neq -h(v)$ follows because the first point has a positive z-coordinate while the second has a negative z-coordinate. The other inequality $h(u) \neq h(v)$ follows because the projections of these points onto the first two coordinates are u and v respectively. This shows that $\overline{h}$ is 1–1. To see that it is onto, recall that we already know this if the third coordinate is zero. But every point on S^2 with nonzero third coordinate is equivalent to one with positive third coordinate, and if $(x, y, z) \in S^2$ with $z > 0$ then simple algebra shows that the point is equal to $h(x, y)$.

Finally, prove that $\mathbb{RP}^2$ is Hausdorff and $\overline{h}$ is a closed mapping.

If the first statement is true, then the second one follows because the domain of $\overline{h}$ is a quotient space of a compact space and continuous maps from compact spaces to Hausdorff spaces are always closed. Since $\overline{h}$ is already known to be continuous, 1–1 and onto, this will prove that it is a homeomorphism.

So how do we prove that $\mathbb{RP}^2$ is Hausdorff? Let v and w be points of S^2 whose images in $\mathbb{RP}^2$ are distinct, and let P_v and P_w be their orthogonal complements in $\mathbb{R}^3$ (hence each is a 2-dimensional vector subspace and a closed subset). Since Euclidean spaces are Hausdorff, we can find an $\varepsilon > 0$ such that $N_\varepsilon(v) \cap P_v = \emptyset$, $N_\varepsilon(w) \cap P_w = \emptyset$, $N_\varepsilon(v) \cap N_\varepsilon(w) = \emptyset$, and $N_\varepsilon(-v) \cap N_\varepsilon(w) = \emptyset$. If T denotes multiplication by -1 on $\mathbb{R}^3$, then these conditions imply that the four open sets

$$N_\varepsilon(v), \quad N_\varepsilon(w), \quad N_\varepsilon(-v) = T\left(N_\varepsilon(v)\right), \quad N_\varepsilon(-w) = T\left(N_\varepsilon(w)\right)$$

are pairwise disjoint. This implies that the images of the distinct points $\pi(v)$ and $\pi(w)$ in $\mathbb{RP}^2$ lie in the disjoint subsets $\pi\,[\,N_\varepsilon(v)\,]$ and $\pi\,[\,N_\varepsilon(w)\,]$ respectively. These are open subsets in $\mathbb{RP}^2$ because their inverse images are given by the open sets $N_\varepsilon(v) \cup N_\varepsilon(-v)$ and $N_\varepsilon(w) \cup N_\varepsilon(-w)$ respectively.■

5. Suppose that X is a topological space with topology $\mathbf{T}$, and suppose also that Y and Z are sets with set-theoretic maps $f : X \to Y$ and $g : Y \to Z$. Prove that the quotient topologies satisfy the condition

$$(g \circ f)_* \mathbf{T} = g_* \left(f_* \mathbf{T}\right) \ .$$

(Informally, a quotient of a quotient is a quotient.)

SOLUTION.

A set W belongs to $(g \circ f)_* \mathbf{T}$ if and only if $(g \circ f)^{-1}[W]$ is open in X. But

$$(g \circ f)^{-1}[W] = f^{-1}\left[g^{-1}[W]\right]$$

so the condition on W holds if and only if $g^{-1}[W]$ belongs to $f_* \mathbf{T}$. The latter in turn holds if and only if w belongs to $g_*(f_* \mathbf{T})$.■

6. If Y is a topological space with a topology $\mathbf{T}$ and $f; X \to Y$ is a set-theoretic map, then the *induced topology* $f^* \mathbf{T}$ on X is defined to be the set of all subsets $W \subset X$ having the form $f^{-1}[U]$ for some open set $U \in \mathbf{T}$. Prove that $f^* \mathbf{T}$ defines a topology on X, that it is the unique smallest topology on X for which f is continuous, and that if $h : Z \to X$ is another set-theoretic map then

$$(f \circ h)^* \mathbf{T} = h^*(f^* \mathbf{T}) \ .$$

SOLUTION.

The object on the left hand side is the family of all sets having the form $(f \circ h)^{-1}[V]$ where V belongs to $\mathbf{T}$. As in the preceding exercise we have

$$(f \circ h)^{-1}[V] = h^{-1}\left[f^{-1}[V]\right]$$

so the family in question is just $h^*(f^* \mathbf{T})$.■

7. Let X and Y be topological spaces, and define an equivalence relation $\mathcal{R}$ on $X \times Y$ by $(x, y) \sim (x', y')$ if and only if $x = x'$. Show that $X \times Y/\mathcal{R}$ is homeomorphic to X.

SOLUTION.

Let $p : X \times Y \to X$ be projection onto the first coordinate. Then $u\mathcal{R}v$ implies $p(u) = p(v)$ and therefore there is a unique continuous map $X \times Y/\mathcal{R} \to X$ sending the equivalence class of (x, y) to x. Set-theoretic considerations imply this map is 1–1 and onto, and it is a homeomorphism because p is an open mapping.■

8. Let $\mathcal{R}$ be an equivalence relation on a topological space X, let $\Gamma_{\mathcal{R}}$ be the graph of $\mathcal{R}$, and let $\pi : X \to X/\mathcal{R}$ be the quotient projection. Prove the following statements:

(a) If $X/\mathcal{R}$ satisfies the Hausdorff Separation Property then $\Gamma_{\mathcal{R}}$ is closed in $X \times X$.

SOLUTION.

If $X/\mathcal{R}$ is Hausdorff then the diagonal $\Delta(X/\mathcal{R})$ is a closed subset of $(X/\mathcal{R}) \times (X/\mathcal{R})$. But $\pi \times \pi$ is continuous, and therefore the inverse image of $\Delta(X/\mathcal{R})$ must be a closed subset of $X \times X$. But this set is simply the graph of $\mathcal{R}$.■

(b) If $\Gamma_{\mathcal{R}}$ is closed and π is open, then $X/\mathcal{R}$ is Hausdorff.

SOLUTION.

If π is open then so is $pi \times \pi$, for the openness of π implies that $\pi \times \pi$ takes basic open subsets of $X \times X$ into open subsets of $(X/\mathcal{R}) \times (X/\mathcal{R})$. By hypothesis the complementary set $X \times X - \Gamma_{\mathcal{R}}$ is open in $X \times X$, and therefore its image, which is

$$(X/\mathcal{R}) \times (X/\mathcal{R}) \ - \ \Delta(X/\mathcal{R})$$

must be open in $(X/\mathcal{R}) \times (X/\mathcal{R})$. But this means that the diagonal $\Delta(X/\mathcal{R})$ must be a closed subset of $(X/\mathcal{R}) \times (X/\mathcal{R})$ and therefore that $X/\mathcal{R}$ must satisfy the Hausdorff Separation Property.■

(c) If $\Gamma_{\mathcal{R}}$ is open then $X/\mathcal{R}$ is discrete.

SOLUTION.

The condition on $\Gamma_{\mathcal{R}}$ implies that each equivalence class is open. But this means that each point in $X/\mathcal{R}$ must be open and hence the latter must be discrete.■

V.2 : Sums and cutting and pasting

Additional exercises

1. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Prove that X is locally connected if and only if each A_α is locally connected.

SOLUTION.

($\Longrightarrow$) If X is locally connected then so is every open subset. But each A_α is an open subset, so each is locally connected.■

($\Longleftarrow$) We need to show that for each $x \in X$ and each open set U containing x there is an open subset $V \subset U$ such that $x \in V$ and V is connected. There is a unique α such that $x = i_\alpha(a)$ for some $a \in A_\alpha$. Let $U_0 = i_\alpha^{-1}(U)$. Then by the local connectedness of A_α and the openness of U_0 there is an open connected set V_0 such that $x \in V_0 \subset U_0$. If $V = i_\alpha(V_0)$, then V has the required properties.■

2. In the preceding exercise, formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be compact.

SOLUTION.

X is compact if and only if each A_α is compact and there are only finitely many (nonempty) subsets in the collection.

The ($\Longrightarrow$) implication follows because each A_α is an open and closed subspace of the compact space X and hence compact, and the only way that the open covering $\{ A_\alpha \}$ of X, which consists of pairwise disjoint subsets, can have a finite subcovering is if it contains only finitely many subsets. To prove the reverse implication, one need only use a previous exercise which shows that a finite union of compact subspaces is compact.■

3. Prove that $\mathbb{RP}^2$ can be constructed by identifying the edge of a Möbius strip with the edge circle on a closed 2-dimensional disk by filling in the details of the following argument: Let $A \subset S^2$ be the set of all points $(x, y, z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T[A] = A$ and $T[B] = B$ so that each of A and B (as well as their intersection) can be viewed as a union of equivalence classes for the equivalence relation that produces $\mathbb{RP}^2$. By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where $\text{sign}(z) = \pm 1$, and thus it follows that the image of B in the quotient space is homeomorphic to $B_+ \cong D^2$. Now consider A. There is a homeomorphism h from $S^1 \times [-1, 1]$ to A sending (x, y, t) to $(\alpha(t)x,\ \alpha(t)y,\ \frac{1}{2}t)$ where

$$\alpha(t) \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

and by construction $h(-v) = -h(v)$. The image of A in the quotient space is thus the quotient of $S^1 \times [-1,1]$ modulo the equivalence relation $u \sim v \iff u = \pm v$. This quotient space is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1,0,t)$ equivalent to $(1,0,-t)$, which yields the Möbius strip. The intersection of this subset in the quotient with the image of B is just the image of the closed curve on the edge of B_+, which also represents the edge curve on the Möbius strip.

FURTHER DETAILS.

We shall fill in some of the reasons that were left unstated in the sketch given above.

Let $A \subset S^2$ be the set of all points $(x,y,z) \in S^2$ such that $|z| \leq \frac{1}{2}$, and let B be the set of all points where $|z| \geq \frac{1}{2}$. If $T(x) = -x$, then $T[A] = A$ and $T(B) = B$ [etc.]

This is true because if $T(v) = w$, then the third coordinates of both points have the same absolute values and of course they satisfy the same inequality relation with respect to $\frac{1}{2}$.

By construction B is a disjoint union of two pieces $B_\pm$ consisting of all points where sign$(z) = \pm 1$,

This is true the third coordinates of all points in B are nonzero.

There is a homeomorphism h from $S^1 \times [-1,1]$ to A sending (x,y,t) to $(\alpha(t)x, \alpha(t)y, \frac{1}{2}t)$ where

$$\alpha(t)s \quad = \quad \sqrt{1 - \frac{t^2}{4}}$$

One needs to verify that h is 1–1 onto; this is essentially an exercise in algebra. Since we are dealing with compact Hausdorff spaces, continuous mappings that are 1–1 onto are automatically homeomorphisms.

This quotient space $[S^1 \times [-1,1]$ modulo the equivalence relation $u \sim v \iff u = \pm v]$ is in turn homeomorphic to the quotient space of the upper semicircular arc S^1_+ (all points with nonnegative y-coordinate) modulo the equivalence relation generated by setting $(-1,0,t)$ equivalent to $(1,0,-t)$, which yields the Möbius strip.

Let $\mathcal{A}$ and $\mathcal{B}$ be the respective equivalence relations on $S^1_+ \times [-1,1]$ and $S^1 \times [-1,1]$, and let $\mathbf{A}$ and $\mathbf{B}$ be the respective quotient spaces. By construction the inclusion $S^1_+ \times [-1,1] \subset S^1 \times [-1,1]$ passes to a continuous map of quotients, and it is necessary and sufficient to check that this map is 1–1 and onto. This is similar to a previous exercise. Points in $S^1 - S^1_+$ all have negative second coordinates and are equivalent to unique points with positive second coordinates. This implies that the mapping from $\mathbf{A}$ to $\mathbf{B}$ is 1–1 and onto at all points except perhaps those whose second coordinates are zero. For such points the equivalence relations given by $\mathcal{A}$ and $\mathcal{B}$ are identical, and therefore the mapping from $\mathbf{A}$ to $\mathbf{B}$ is also 1–1 and onto at all remaining points.■

4. Suppose that the topological space X is a union of two closed subspaces A and B, let $C = A \cap B$, let $h : C \to C$ be a homeomorphism, and let $A \cup_h B$ be the space formed from $A \sqcup B$ by identifying $x \in C \subset A$ with $h(x) \in C \subset B$. Prove that $A \cup_h B$ is homeomorphic to X if h extends to a homeomorphism $H : A \to A$, and give an example for which X is not homeomorphic to $A \cup_h B$. [*Hint:* Construct the homeomorphism using H in the first case, and consider also the case where $X = S^1 \sqcup S^1$, with $A_\pm == S^1_\pm \sqcup S^1_\pm$; then $C = \{\pm 1\} \times \{1,2\}$, and there is a homeomorphism from h to itself such that $A_+ \cup_h A_-$ is connected.]

SOLUTION.

We can and shall view X as $A \cup_{\mathrm{id}} B$.

Consider the map $F_0 : A \sqcup B \to A \sqcup B$ defined by H^{-1} on A and the identity on B. We claim that this passes to a unique continuous map of quotients from X to $A \cup_h B$; *i.e.*, the map F_0 sends each nonatomic equivalence classes $\{ (c,1), (c,2) \}$ for $X = A \cup_{\mathrm{id}} B$ to a nonatomic equivalence class of the form $\{ (u,1), (h(u),2) \}$ for $A \cup_h B$. Since F_0 sends $(c,1)$ to $(h^{-1}(c),1)$ and $(c,2)$ to itself, we can verify the compatibility of F_0 with the equivalence relations by taking $u = h^{-1}(c)$. Passage to the quotients then yields the desired map $F : X \to A \cup_h B$.

To show this map is a homeomorphism, it suffices to define Specifically, start with $G_0 = F_0^{-1}$, so that $G_0 = H$ on A and the identity on B. In this case it is necessary to show that a nonatomic equivalence class of the form $\{ (u,1), (h(u),2) \}$ for $A \cup_h B$ gets sent to a nonatomic equivalence class of the form $\{ (c,1), (c,2) \}$ for $X = A \cup_{\mathrm{id}} B$. Since G_0 maps the first set to $\{ (h(u),1), (h(u),2) \}$ this is indeed the case, and therefore G_0 also passes to a map of quotients which we shall call G.

Finally we need to verify that F and G are inverses to each other. By construction the maps F_0 and G_0 satisfy $F([y]) = [F_0(y)]$ and $G([z]) = [G_0(z)]$, where square brackets denote equivalence classes. Therefore we have

$$G \circ F([y]) = G\left([F_0(y)]\right) = [G_0\left(F_0(y)\right)]$$

which is equal to $[y]$ because F_0 and G_0 are inverse to each other. Therefore $G \circ F$ is the identity on X. A similar argument shows that $F \circ G$ is the identity on $A \cup_h B$.

To construct the example where X is **not** homeomorphic to $A \cup_h B$, we follow the hint and try to find a homeomorphism of the four point space $\{\pm 1\} \times \{1,2\}$ to itself such that X is **not** homeomorphic to $A \cup_h B$ is connected; this suffices because we know that X is not connected. Sketches on paper or physical experimentation with wires or string are helpful in finding the right formula.

Specifically, the homeomorphism we want is given as follows:

$$\begin{array}{rcl} (-1,1) \in A_+ & \longrightarrow & (1,2) \in A_- \\ (1,1) \in A_+ & \longrightarrow & (1,1) \in A_- \\ (1,2) \in A_+ & \longrightarrow & (-1,1) \in A_- \\ (-1,2) \in A_+ & \longrightarrow & (-1,2) \in A_- \end{array}$$

The first of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{1\}$ lie in the same component of the quotient space, the second of these implies that the images of $S^1_- \times \{1\}$ and $S^1_+ \times \{1\}$ both lie in the same component, and the third of these implies that the images of $S^1_+ \times \{2\}$ and $S^1_- \times \{2\}$ also lie in the same component. Since the entire space is the union of the images of the connected subsets $S^1_\pm \times \{1\}$ and $S^1_\pm \times \{2\}$ it follows that $A \cup_h B$ is connected.■

FOOTNOTE.

The argument in the first part of the exercise remains valid if A and B are open rather than closed subsets.■

5. *One-point unions.* One conceptual problem with the disjoint union of topological spaces is that it is never connected except for the trivial case of one summand. In many geometrical and topological contexts it is extremely useful to construct a modified version of disjoint unions that is connected if all the pieces are. Usually some additional structure is needed in order to make such constructions.

In this exercise we shall describe such a construction for objects known as *pointed spaces* that are indispensable for many purposes (*e.g.*, the definition of fundamental groups as in Munkres). A pointed space is a pair (X, x) consisting of a topological space X and a point $x \in X$; we often call x the *base point*, and unless stated otherwise *the one point subset consisting of the base point is assumed to be closed.* If (Y, y) is another pointed space and $f : X \to Y$ is continuous, we shall say that *f is a base point preserving continuous map from (X, x) to (Y, y)* if $f(x) = y$, In this case we shall often write $f : (X, x) \to (Y, y)$. Identity maps are base point preserving, and composites of base point preserving maps are also base point preserving.

Given a finite collection of pointed spaces (X_i, x_i), define an equivalence relation on $\coprod_i X_i$ whose equivalence classes consist of $\coprod_j \{x_j\}$ and all one point sets y such that $y \notin \coprod_j \{x_j\}$. Define the *one point union* or *wedge*

$$\bigvee_{i=1}^{n} (X_j, x_j) \quad = \quad (X_1, x_1) \vee \cdots \vee (X_n, x_n)$$

to be the quotient space of this equivalence relation with the quotient topology. The base point of this space is taken to be the class of $\coprod_j \{x_j\}$.

(a) Prove that the wedge is a union of closed subspaces Y_j such that each Y_j is homeomorphic to X_j and if $j \neq k$ then $Y_j \cap Y_k$ is the base point. Explain why $\vee_k\, (X_k, x_k)$ is Hausdorff if and only if each X_j is Hausdorff, why $\vee_k\, (X_k, x_k)$ is compact if and only if each X_j is compact, and why $\vee_k\, (X_k, x_k)$ is connected if and only if each X_j is connected (and the same holds for arcwise connectedness).

SOLUTION.

For each j let $\mathbf{in}_j : X_j \to \coprod_k X_k$ be the standard injection into the disjoint union, and let

$$P : \coprod_k X_k \longrightarrow \bigvee_k (X_k, x_k)$$

be the quotient map defining the wedge. Define Y_j to be $P \circ \mathbf{in}_j[\,X_j\,]$. By construction the map $P \circ \mathbf{in}_j$ is continuous and 1–1; we claim it also sends closed subsets of X_j to closed subsets of the wedge. Suppose that $F \subset X_j$ is closed; then $P \circ \mathbf{in}_j[F]$ is closed in the wedge if and only if its inverse image under P is closed. But this inverse image is the union of the closed subsets $\mathbf{in}_j[F]$ and $\coprod_k \{x_k\}$ (which is a finite union of one point subsets that are assumed to be closed). It follows that Y_j is homeomorphic to $X + j$. The condition on $Y_k \cap Y_\ell$ for $k \neq \ell$ is an immediate consequence of the construction.

The assertion that the wedge is Hausdorff if and only if each summand is follows because a subspace of a Hausdorff space is Hausdorff, and a finite union of closed Hausdorff subspaces is always Hausdorff (by a previous exercise).

To verify the assertions about compactness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\, (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is compact, then its continuous under q_j, which is the image of X_j, must also be compact. Conversely if the sets X_j are compact for all j, then the (finite!) union of their images, which is the entire wedge, must be compact.

To verify the assertions about connectedness, note first that for each j there is a continuous collapsing map q_j from $\vee_k\, (X_k, x_k)$ to (X_j, x_j), defined by the identity on the image of (X_j, x_j) and by sending everything to the base point on every other summand. If the whole wedge is connected, then its continuous under q_j, which is the image of X_j, must also be connected. Conversely if the sets X_j are connected for all j, then the union of their images, which is the entire wedge, must be connected because all these images contain the base point. Similar statements hold for arcwise connectedness and follow by inserting "arcwise" in front of "connected" at every step of the argument.∎

(b) Let $\varphi_j : (X_j, x_j) \to \vee_k\ (X_k, x_k)$ be the composite of the injection $X_j \to \coprod_k\ X_k$ with the quotient projection; by construction φ_j is base point preserving. Suppose that (Y, y) is some arbitrary pointed space and we are given a sequence of base point preserving continuous maps $F_j : (X_j, x_j) \to (Y, y)$. Prove that there is a unique base point preserving continuous mapping

$$F : \vee_k\ (X_k, x_k) \to (Y, y)$$

such that $F \circ \varphi_j = F_j$ for all j.

SOLUTION.

To prove existence, first observe that there is a unique continuous map $\widetilde{F} : \coprod_k X_k \to Y$ such that $\mathbf{in}_j \circ \widetilde{F} = F_j$ for all j. This passes to a unique continuous map F on the quotient space $\vee_k\ (X_k, x_k)$ because $\widetilde{F}$ is constant on the equivalence classes associated to the quotient projection P. This constructs the map we want; uniqueness follows because the conditions prescribe the definition at every point of the wedge.■

(c) In the infinite case one can carry out the set-theoretic construction as above but some care is needed in defining the topology. Show that if each X_j is Hausdorff and one takes the so-called *weak topology* whose closed subsets are generated by the family of subsets $\varphi_j[F]$ where F is closed in X_j for some j, then [1] a function h from the wedge into some other space Y is continuous if and only if each composite $h \circ \varphi_j$ is continuous, [2] the existence and uniqueness theorem for mappings from the wedge (in the previous portion of the exercise) generalizes to infinite wedges with the so-called weak topologies.

SOLUTION.

Strictly speaking, one should verify that the so-called weak topology is indeed a topology on the wedge. We shall leave this to the reader.

To prove [1], note that ($\Longrightarrow$) is trivial. For the reverse direction, we need to show that if E is closed in Y then $h^{-1}[E]$ is closed with respect to the so-called weak topology we have defined. The subset in question is closed with respect to this topology if and only if $h^{-1}[E] \cap \varphi[X_j]$ is closed in $\varphi[X_j]$ for all j, and since φ_j maps its domain homeomorphically onto its image, the latter is true if and only if $\varphi^{-1} \circ h^{-1}[E]$ is closed in X_j for all j. But these conditions hold because each of the maps $\varphi_j \circ h$ is continuous. To prove [2], note first that there is a unique set-theoretic map, and then use [1] to conclude that it is continuous.■

(d) Suppose that we are given an infinite wedge such that each summand is Hausdorff and contains at least two points. Prove that the wedge with the so-called weak topology is not compact.

SOLUTION.

For each j let $y_j \in X_j$ be a point other than x_j, and consider the set E of all points y_j. This is a closed subset of the wedge because its intersection with each set $\varphi[X_j]$ is a one point subset and hence closed. In fact, every subset of E is also closed by a similar argument (the intersections with the summands are either empty or contain only one point), so E is a discrete closed subset of the wedge. Compact spaces do not have infinite discrete closed subspaces, and therefore it follows that the infinite wedge with the weak topology is not compact.■

Remark. If each of the summands in **(d)** is compact Hausdorff, then there is a natural candidate for a *strong topology* on a countably infinite wedge which makes the latter into a compact Hausdorff space. In some cases this topology can be viewed more geometrically; for example, if each (X_j, x_j)

is equal to $(S^1, 1)$ and there are countably infinitely many of them, then the space one obtains is the *Hawaiian earring* in $\mathbb{R}^2$ given by the union of the circles defined by the equations

$$\left(x - \frac{1}{2^k}\right)^2 \; + \; + \, y^2 \quad = \quad \frac{1}{2^{2k}} \; .$$

As usual, drawing a picture may be helpful. The $k^{\rm th}$ circle has center $(1/2^k,\, 0)$ and passes through the origin; the y-axis is the tangent line to each circle at the origin.

SKETCHES OF VERIFICATIONS OF ASSERTIONS.

If we are given an infinite sequence of compact Hausdorff pointed spaces $\{\, (X_n, x_n)\,\}$ we can put a compact Hausdorff topology on their wedge as follows. Let W_k be the wedge of the first k spaces; then for each k there is a continuous map

$$q_k : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad W_k$$

(with the so-called weak topology on the wedge) that is the identity on the first k summands and collapses the remaining ones to the base point. These maps are in turn define a continuous function

$$\mathbf{q} : \bigvee_n (X_n, x_n) \quad \longrightarrow \quad \prod_k W_k$$

whose projection onto W_k is q_k. This mapping is continuous and 1–1; if its image is closed in the (compact!) product topology, then this defines a compact Hausdorff topology on the infinite wedge $\vee_n (X_n, x_n)$.

Here is one way of verifying that the image is closed. For each k let $c_k : W_k \to W_{k-1}$ be the map that is the identity on the first $(k-1)$ summands and collapses the last one to a point. Then we may define a continuous map C on $\prod_{k\geq 1} W_k$ by first projecting onto the product $\prod_{k\geq 2} W_k$ (forget the first factor) and then forming the map $\prod_{k\geq 2} W_k$. The image of $\mathbf{q}$ turns out to be the set of all points $\mathbf{x}$ in the product such that $C(\mathbf{x}) = \mathbf{x}$. Since the product is Hausdorff the image set is closed in the product and thus compact.

A comment about the compactness of the Hawaiian earring E might be useful. Let F_k be the union of the circles of radius 2^{-j} that are contained in E, where $j \leq k$, together with the closed disk bounded by the circle of radius $2^{-(k+1)}$ in E. Then F_k is certainly closed and compact. Since E is the intersection of all the sets F_k it follows that E is also closed and compact.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 5

Fall 2008

VI. Spaces with additional properties

VI.1 : Second countable spaces

Problems from Munkres, § 30, pp. 194 – 195

9. [*First part only*] *Let X be a Lindelöf space, and suppose that A is a closed subset of X. Prove that A is Lindelöf.*

SOLUTION.

The statement and proof are parallel to a result about compact spaces in the course notes, the only change being that "compact" is replaced by "Lindelöf."■

10. Show that if X is a countable product of spaces having countable dense subsets, then X also has a countable dense subset.

SOLUTION.

Suppose first that X is a finite product of spaces Y_i such that each Y_i has a countable dense subset D_i. Then $\prod_i D_i$ is countable and

$$X \;=\; \prod_i Y_i \;=\; \prod_i \overline{D_i} \;=\; \overline{\prod_\alpha D_i}\ .$$

Suppose now that X is countably infinite. The same formula holds, but the product of the D_i's is not necessarily countable. To adjust for this, pick some point $\delta_j \in D_j$ for each j and consider the set E of all points $(a_0, a_1, \cdots)$ in $\prod_j D_j$ such that $a_j = \delta_j$ for all but at most finitely many values of j. This set is countable, and we claim it is dense. It suffices to show that every basic open subset contains at least one point of E. But suppose we are given such a set $V = \prod_j V_j$ where V_j is open in X_j and $V_j = X_j$ for all but finitely many j; for the sake of definiteness, suppose this happens for $j > M$. For $j \leq M$, let $b_j \in D_j \cap V_j$; such a point can be found since D_j is dense in X_j. Set $b_j = \delta_j$ for $j > M$. If we let $b = (b_0, b_1, \cdots)$, then it then follows that $b \in E \cap V$, and this implies E is dense in the product.■

13. Show that if X has a countable dense subset, then every collection of disjoint open subsets in X is countable.

SOLUTION.

This is similar to the proof that an open subset of $\mathbb{R}^n$ has only countably many components.■

14. Show that if X is compact and Y is Lindelöf, then $X \times Y$ is Lindelöf.

SOLUTION.

This is essentially the same argument as the one showing that a product of two compact spaces is compact. The only difference is that after one constructs an open covering of Y at one step in the proof, then one only has a countable subcovering and this leads to the existence of a countable subcovering of the product. Details are left to the reader.■

Additional exercises

1. If $(X, \mathbf{T})$ is a second countable Hausdorff space, prove that the cardinalities of both X and $\mathbf{T}$ are less than or equal to $2^{\aleph_0}$. (Using the formulas for cardinal numbers in Section I.3 of the course notes and the separability of X one can prove a similar inequality for $\mathbf{BC}(X)$.)

SOLUTION.

List the basic open subsets in $\mathcal{B}$ as a sequence $U_0, U_1, U_2, \cdots$ If W is an open subset of U define a function $\psi_W : \mathbb{N} \to \{0,1\}$ by $\psi_W(i) = 1$ if $U_i \subset W$ and 0 otherwise. Since $\mathcal{B}$ is a basis, every open set W is the union of the sets U_i for which $\psi_W(i) = 1$. In particular, the latter implies that if $\psi_V = \psi_W$ then $V = W$ and hence we have a 1–1 map from the set of all open subsets to the set of functions from $\mathbb{N}$ to $\{0,1\}$. Therefore the cardinality of the family of open subsets is at most the cardinality of the set of functions, which is $2^{\aleph_0}$.

If X is Hausdorff, or even if we only know that every one point subset of X is closed in X, then we may associate to each $x \in X$ the open subset $X - \{x\}$. If $x \neq y$ then $X - \{x\} \neq X - \{y\}$ and therefore the map $C : X \to \mathbf{T}$ defined in this fashion is 1–1. Therefore we have $|X| \leq |\mathbf{T}|$, and therefore by the preceding paragraph we know that $|X| \leq 2^{\aleph_0}$.■

A somewhat more complicated argument yields similar conclusions for spaces satisfying the weaker $\mathbf{T_0}$ condition stated in Section VI.3.■

2. *Separability and subspaces.* The following example shows that a closed subspace of a separable Hausdorff space is not necessarily separable.

(a) Let X be the upper half plane $\mathbb{R} \times [0, \infty)$ and take the topology generated by the usual metric topology plus the following sets:

$$T_\varepsilon(x) \quad = \quad \{\,(x,0)\,\} \;\cup\; N_\varepsilon\,(\,(x,\varepsilon)\,)\,, \quad \text{where} \quad x \in \mathbb{R} \quad \text{and} \quad \varepsilon \;>\; 0\,\}$$

Geometrically, one takes the interior region of the circle in the upper half plane that is tangent to the x-axis at $(x, 0)$ and adds the point of tangency. — Show that the x-axis is a closed subset and has the discrete topology.

SOLUTION.

The x-axis is closed because it is closed in the ordinary Euclidean topology and the "new" topology contains the Euclidean topology; therefore the x-axis is closed in the "new" topology. The subspace topology on the x-axis is the discrete topology intersection of the open set $T_\varepsilon(x)$ with the real axis is $\{\,x\,\}$.■

(b) Explain why the space in question is Hausdorff. [*Hint:* The topology contains the metric topology. If a topological space is Hausdorff and we take a larger topology, why is the new topology Hausdorff?]

SOLUTION.

In general if $(X, \mathbf{T})$ is Hausdorff and $\mathbf{T} \subset \mathbf{T}^*$ then $(X, \mathbf{T}^*)$ is also Hausdorff, for a pair of disjoint $\mathbf{T}$-open subsets containing distinct point u and v will also be a pair of $\mathbf{T}$-open subsets with the same properties.■

(c) Show that the set of points (u, v) in X with $v > 0$ and $u, v \in \mathbb{Q}$ is dense. [*Hint:* Reduce this to showing that one can find such a point in every set of the form $T_\varepsilon(x)$.]

SOLUTION.

To show that a subset D of a topological space X is dense, it suffices to show that the intersection of D with every nonempty open subset in some base $\mathcal{B}$ is nonempty (why?). Thus we need to show that the generating set is a base for the topology and that every basic open subset contains a point with rational coordinates.

The crucial point to showing that we have a base for the topology is to check that if U and V are basic open subsets containing some point $z = (x, y)$ then there is some open subset in the generating collection that contains z and is contained in $U \cap V$. If U and V are both basic open subsets we already know this, while if $z \in T_\varepsilon(a) \cap T_\delta(b)$ there are two cases depending upon whether or not z lies on the x-axis; denote the latter by A. If $z \notin A$ then it lies in the metrically open subsets $T_\varepsilon(a) - A$ and $T_\varepsilon(b) - A$, and one can find a metrically open subset that contains z and is contained in the intersection. On the other hand, if $z \in A$, then $T_\varepsilon(a) \cap A = \{a\}$ and $T_\delta(b) \cap A = \{b\}$ imply $a = b$, and the condition on intersections is immediate because the intersection of the subsets is either $T_\varepsilon(a)$ or $T_\delta(a)$ depending upon whether $\delta \leq \varepsilon$ or vice versa.■

3. Let $\{A_\alpha \mid \alpha \in \mathcal{A}\}$ be a family of topological spaces, and let $X = \coprod_\alpha A_\alpha$. Formulate and prove necessary and sufficient conditions on $\mathcal{A}$ and the sets A_α for the space X to be second countable, separable or Lindelöf.

SOLUTION.

For each property $\mathcal{P}$ given in the exercise, the space X has property $\mathcal{P}$ if and only if each A_α does and there are only finitely many α for which A_α is nonempty. The verifications for the separate cases are different and will be given in reverse sequence.

The Lindelöf property.

The proof in this case is the same as the proof we gave for compactness in an earlier exercises with "countable" replacing "finite" throughout.■

Separability.

($\Longrightarrow$) Let D be the countable dense subset. Each A_α must contain some point of D, and by construction this point is not contained in any of the remaining sets A_β. Thus we have a 1–1 function from $\mathcal{A}$ to D sending α to a point $d(\alpha) \in A_\alpha \cap D$. This implies that the cardinality of $\mathcal{A}$ is at most $|D| \leq \aleph_0$.■

($\Longleftarrow$) If D_α is a dense subset of A_α and $\mathcal{A}$ is countable, then $\cup_\alpha D_\alpha$ is a countable dense subset of $\mathcal{A}$.■

Second countability.

($\Longrightarrow$) Since a subspace of a second countable space is second countable, each A_α must be second countable. Since the latter condition implies both separability and the Lindelöf property, the preceding arguments show that only countably many summands can be nontrivial.

($\Longleftarrow$) If $\mathcal{A}$ is countable and $\mathcal{B}_\alpha$ is a countable base for A_α then $\cup_\alpha \mathcal{B}_\alpha$ determines a countable base for X (work out the details!).■

VI.2 : Compact spaces – II

Problems from Munkres, § 28, pp. 181 – 182

6. Prove that if $(X, \mathbf{d})$ is a metric space and $f : X \to X$ is a distance preserving map (an *isometry*), then f is $1-1$ onto and hence a homeomorphism. [*Hint:* If $a \notin f(X)$ choose $\varepsilon > 0$ so that the ε-neighborhood of a is disjoint from $f[X]$. Set $x_0 = a$ and $x_{n+1} = f(x_n)$ in general. Show that $d(x_n, x_m) \geq \varepsilon$ for $n \neq m$.]

SOLUTION.

Follow the hint to define a and the sequence. The existence of ε is guaranteed because $a \notin f[X]$ and the compactness of $f[X]$ imply that the continuous function $g(x) = \mathbf{d}(a, f(x))$ is positive valued, and it is bounded away from 0 because it attains a minimum value. Since $x_k \in f[X]$ for all k it follows that $\mathbf{d}(a, x_k) > \varepsilon$ for all k. Given $n \neq m$ write $m = n + k$; reversing the roles of m and n if necessary we can assume that $k > 0$. If f is distance preserving, then so is every n-fold iterated composite $\circ^n f$ of f with itself. Therefore we have that

$$\mathbf{d}(x_n, x_m) \quad = \quad \mathbf{d}\left(\circ^n f(a), \circ^n f(x_k)\right) \quad = \quad \mathbf{d}(a, x_k) \quad > \quad \varepsilon$$

for all distinct nonnegative integers m and n. But this cannot happen if X is compact, because the latter implies that $\{ x_n \}$ has a convergent subsequence. This contradiction implies that our original assumption about the existence of a point $a \notin f[X]$ is false, so if is onto. On the other hand $x \neq y$ implies

$$0 \quad < \quad \mathbf{d}(x, y) \quad = \quad \mathbf{d}\left(f(x), f(y) \right)$$

and thus that f is also 1–1. Previous results now also imply that f is a homeomorphism onto its image.■

FOOTNOTE.

To see the need for compactness rather than (say) completeness, consider the map $f(x) = x+1$ on the set $[0, +\infty)$ of nonnegative real numbers.■

Additional exercises

1. Let X be a compact Hausdorff space, let Y be a Hausdorff space, and let $f : X \to Y$ be a continuous map such that f is locally 1–1 (each point x has a neighborhood U_x such that $f|U_x$ is 1–1) and there is a closed subset $A \subset X$ such that $f|A$ is 1–1. Prove that there is an open neighborhood V of A such that $f|V$ is 1–1. [*Hint:* A map g is 1–1 on a subset B is and only if

$$B \times B \cap (g \times g)^{-1}[\,\Delta_Y)\,] \quad = \quad \Delta_B$$

where Δ_S denotes the diagonal in $S \times S$. In the setting of the exercise show that

$$(f \times f)^{-1}[\,\Delta_Y\,] \quad = \quad \Delta_X \cup D',$$

where D' is closed and disjoint from the diagonal. Also show that the subsets D' and $A \times A$ are disjoint, and find a square neighborhood of $A \times A$ disjoint from D'.]

SOLUTION.

Follow the steps in the hint.

> *A map g is $1-1$ on a subset B is and only if $B \times B \cap (g\times g)^{-1}[\,\Delta_Y\,] = \Delta_B$ where Δ_S denotes the diagonal in $S \times S$.*

This is true because the set on the left hand side of the set-theoretic equation is the set of all (b, b') such that $g(b) = g(b')$. If there is a nondiagonal point in this set then the function is not 1–1, and conversely if the function is 1–1 then there cannot be any off-diagonal terms in the set.■

> *Show that if $D' = (f\times f)^{-1}[\,\Delta_Y\,] - \Delta_X$, then D' is closed and disjoint from the diagonal.*

Since f is locally 1–1, for each $x \in X$ there is an open set U_x such that $f|U_x$ is 1–1; by the first step we have

$$U_x \times U_x \cap (f\times f)^{-1}[\,\Delta_Y\,] \quad = \quad \Delta_{U_x} \ .$$

If we set $W = \cap_x \, U_x \times U_x$ then W is an open neighborhood of Δ_X and by construction we have

$$(f\times f)^{-1}[\,\Delta_Y\,] \ \cap \ W \quad = \quad \Delta_X$$

which shows that Δ_X is open in $(f \times f)^{-1}[\,\Delta_Y\,]$ and thus its relative complement in the latter — which is D' — must be closed.■

> *Show that the subsets D' and $A\times A$ are disjoint, and find a square neighborhood of $A\times A$ disjoint from D'.*

Since $f|A$ is 1–1 we have $A\times A\cap(f\times f)^{-1}[\,\Delta_Y\,] = \Delta_A$, which lies in $\Delta_X = (f\times f)^{-1}[\,\Delta_Y\,] - D'$. Since A is a compact subset of the open set $X \times X - D'$, by Wallace's Theorem there is an open set U such that

$$A\times A \quad \subset \quad U\times U \quad \subset \quad X\times X - D' \ .$$

The first part of the proof now implies that $f|U$ is 1–1.■

2. Let U be open in $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a $\mathbf{C}^1$ map such that $Df(x)$ is invertible for all $x \in U$ and there is a compact subset $A \subset U$ such that $f|A$ is $1-1$. Prove that there is an open neighborhood V of A such that $f|V$ is a homeomorphism onto its image.

SOLUTION.

By the Inverse Function Theorem we know that f is locally 1–1, and therefore by the previous exercise we know that f is 1–1 on an open set V such that $A \subset V \subset U$. But the Inverse Function Theorem also implies that f locally has a $\mathbf{C}^1$ inverse on U. Since f has a global set-theoretic inverse from $f[V]$ back to V, it follows that this global inverse is also $\mathbf{C}^1$.■

3. Let $\mathbf{d}_p$ be the metric on the integers constructed in Exercise I.1.1, and let $\widehat{\mathbb{Z}_p}$ be the completion of this metric space. Prove that $\widehat{\mathbb{Z}_p}$ is (sequentially) compact. [*Hint:* For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers. Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a \bmod p^{r+1}$. Now let $\{\, a_n \,\}$ be an infinite sequence of integers, and assume the sequence takes infinitely many distinct values (otherwise the sequence obviously has a convergent subsequence). Find a sequence of positive integers $\{\, b_r \,\}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many points in the sequence and $b_{r+1} \equiv b_r \bmod p^{r+1}$. Form a subsequence of $\{\, a_n \,\}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]

SOLUTION.

As usual, we follow the steps in the hint.

> *For each integer $r > 0$ show that every integer is within p^{-r} of one of the first p^{r+1} nonnegative integers.*

If n is an integer, use the long division property to write $n = p^{r+1}a$ where $0 \leq a < p^{r+1}$. We then have $\mathbf{d}_p(n, a) \leq p^{-(r+1)} < p^r$.■

> *Furthermore, each open neighborhood of radius p^{-r} centered at one of these integers a is a union of p neighborhoods of radius $p^{-(r+1)}$ over all of the first p^{r+2} integers b such that $b \equiv a \bmod p^{r+1}$.*

Suppose we do long division by p^{r+2} rather than p^{r+1} and get a new remainder a'. How is it related to a? Very simply, a' must be one of the p nonnegative integers b such that $0 \leq b < p^{r+2}$ and $b \equiv a \bmod p^{r+1}$. Since $\mathbf{d}(u, v) < p^{-r}$ if and only if $\mathbf{d}_p(u, v,) \leq p^{-(r+1)}$, it follows that $N_{p^{-r}}(a)$ is a union of p open subsets of the form $N_{p^{-(r+1)}}(b)$ as claimed (with the numbers b satisfying the asserted conditions).

> *Find a sequence of nonnegative integers $\{ b_r \}$ such that the open neighborhood of radius p^{-r} centered at b_r contains infinitely many terms in the sequence and $b_{r+1} \equiv b_r \bmod p^{r+1}$.*

This is very similar to a standard proof of the Bolzano-Weierstrass Theorem in real variables. Infinitely many terms of the original sequence must lie in one of the sets $N_1(b_0)$ where $1 \leq b < p$. Using the second step we know there is some b_1 such that $b_1 \equiv b_0 \bmod p$ and infinitely many terms of the sequence lie in $N_{p^{-1}}(b_1)$, and one can continue by induction (fill in the details!).■

> *Form a subsequence of $\{ a_n \}$ by choosing distinct points $a_{n(k)}$ recursively such that $n(k) > n(k-1)$ and $a_{n(k)} \in N_{p^{-k}}(b_k)$. Prove that this subsequence is a Cauchy sequence and hence converges.]*

We know that the neighborhoods in question contain infinitely many values of the sequence, and this allows us to find $n(k)$ recursively. It remains to show that the construction yields a Cauchy sequence. The key to this is to observe that $a_{n(k)} \equiv b_k \bmod p^{k+1}$ and thus we also have

$$\mathbf{d}_p \left(a_{n(k+1)}, a_{n(k)} \right) \quad < \quad \frac{1}{p^k} \ .$$

Similarly, if $\ell > k$ then we have

$$\mathbf{d}_p \left(a_{n(\ell)}, a_{n(k)} \right) \quad < \quad \frac{1}{p^{\ell-1}} \ + \ \cdots \ + \ \frac{1}{p^k} \ .$$

Since the geometric seris $\sum_k p^{-k}$ converges, for every $\varepsilon > 0$ there is an M such that $\ell, k \geq M$ implies the right had side of the displayed inequality is less than ε, and therefore it follows that the constructed subsequence is indeed a Cauchy sequence. By completeness (of the completion) this sequence converges. Therefore $\widehat{\mathbb{Z}_p}$ is (sequentially) compact.■

VI.3 : Separation axioms

Problem from Munkres, § 26, pp. 170 − 172

11. Let X be a compact Hausdorff space, and let $\{A_\alpha\}$ be a family of nonempty closed connected subsets ordered by inclusion. Prove that $Y = \cap_\alpha A_\alpha$ is connected. [*Hint:* If $C \cup D$ is a separation of Y, choose disjoint open sets U and V of X containing C and D respectively and show that

$$B \;=\; \bigcap_\alpha (A_\alpha - (U \cup V))$$

is nonempty.]

SOLUTION.

Follow the suggestion of the hint to define C, D and to find U, V. There are disjoint open subset sets $U, V \subset X$ containing C and D respectively because a compact Hausdorff space is $\mathbf{T_4}$.

For each α the set $B_\alpha = A_\alpha - (U \cup V)$ is a closed and hence compact subset of X. If each of these subsets is nonempty, then the linear ordering condition implies that the family of closed compact subsets B_α has the finite intersection property; specifically, the intersection

$$B_{\alpha(1)} \cap \ \dots \ \cap B_{\alpha(k)}$$

is equal to $B_{\alpha(j)}$ where $A_{\alpha(j)}$ is the smallest subset in the linearly ordered collection

$$\{A_{\alpha(1)}, \ \dots \ , A_{\alpha(k)}\} \ .$$

Therefore by compactness it will follow that the intersection

$$\bigcap_\alpha B_\alpha \;=\; \left(\bigcap_\alpha A_\alpha\right) - (U \cup V)$$

is nonempty. But this contradicts the conditions $\cap_\alpha A_\alpha \;=\; C \cup D \;\subset\; U \cup V$. Therefore it follows that the $\cap_\alpha A_\alpha$ must be connected.

Therefore we only need to answer the following question: Why should the sets $A_\alpha - (U \cup V)$ be nonempty? If the intersection is empty then $A_\alpha \subset U \cup V$. By construction we have $C \subset A_\alpha \cap U$ and $D \subset A_\alpha \cap V$, and therefore we can write A_α as a union of two nonempty disjoint open subsets. However, this contradicts our assumption that A_α is connected and therefore we must have $A_\alpha - (U \cup V) \neq \emptyset$.■

Problems from Munkres, § 33, pp. 212 − 214

2. **(a)** [*For metric spaces.*] Show that a connected metric space having more than one point is uncountable.

SOLUTION.

The proof is based upon Urysohn's Lemma and therefore is valid in arbitrary $\mathbf{T_4}$ spaces; we have stated it only for metric spaces because we have only established Urysohn's Lemma in that case.

Let X be the ambient topological space and suppose that u and v are distinct points of X. Then $\{u\}$ and $\{v\}$ are disjoint closed subsets and therefore there is a continuous function $f : X \to \mathbb{R}$ such that $f(u) = 0$ and $f(v) = 1$. Since $f[X]$ is connected, for each $t \in [0,1]$ there is a point $x_t \in X$ such that $f(x_t) = t$. Since $s \neq t$ implies $f(x_s) \neq f(x_t)$, it follows that the map $x : [0,1] \to X$ sending t to x_t is 1–1. Therefore we have $2^{\aleph_0} \leq |X|$, and hence X is uncountable.■

6. A space X is said to be *perfectly normal* if every closed set in X is a $\mathbf{G}_\delta$ subset of X. [*Note:* The latter are defined in Exercise 1 on page 194 of Munkres and the reason for the terminology is also discussed at the same time.]

(a) Show that every metric space is perfectly normal.

SOLUTION.

Let $(X, \mathbf{d})$ be a metric space, let $A \subset X$ be closed and let $f(x) = \mathbf{d}(x, A)$. Then

$$A \quad = \quad f^{-1}(\{0\}) \quad = \quad \bigcap_n f^{-1}\Big[\left(-\tfrac{1}{n}, \tfrac{1}{n}\right) \Big]$$

presents A as a countable intersection of open subsets.■

8. Let X be a completely regular space, and let A and B be disjoint closed subsets such that A is compact. Prove that there is a continuous function $f : X \to [0,1]$ that is 0 on A and 1 on B.

SOLUTION.

For each $a \in A$ there is a continuous function $f_a : X \to [-1,1]$ that is -1 at a and 0 on B. Let $U_a = f_a^{-1}[[-1,0)]$. Then the sets U_a define an open covering of A and hence there is a finite subcovering corresponding to $U_{a(1)}, \cdots, U_{a(k)}$. Let f_i be the function associated to $a(i)$, let g_i be the maximum of f_i and 0 (so g_i is continuous by a previous exercise), and define

$$f \quad = \quad \prod_{i=1}^{k} g_k \ .$$

By construction the value of f is 1 on B because each factor is 1 on B, and $f = 0$ on $\cup_i U_{a(i)}$ because $g_j = 0$ on $U_{a(j)}$; since the union contains A, it follows that $f = 0$ on A.■

Additional exercises

1. If $(X, \mathbf{T})$ is compact Hausdorff and $\mathbf{T}^*$ is strictly contained in $\mathbf{T}$, prove that $(X, \mathbf{T}^*)$ is compact but not Hausdorff.

SOLUTION.

The strict containment condition implies that the identity map from $(X, \mathbf{T})$ to $(X, \mathbf{T}^*)$ is continuous but not a homeomorphis. Since the image of a compact set is compact, it follows that $(X, \mathbf{T}^*)$ is compact. If it were Hausdorff, the identity map would be closed and thus a homeomorphism. Therefore $(X, \mathbf{T}^*)$ is not Hausdorff.■

2. **(a)** Prove that a topological space is $\mathbf{T_3}$ if and only if it is $\mathbf{T_1}$ and there is a basis $\mathcal{B}$ such that for every $x \in X$ and every open set $V \in \mathcal{B}$ containing x, there is an open subset $W \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset V$.

SOLUTION.

$(\Longrightarrow)$ Suppose that X is $\mathbf{T_3}$, let $x \in X$ and let V be a basic open subset containing x. Then there is an open set U in X such that $x \in U \subset \overline{U} \subset V$. Sinc $\mathcal{B}$ is a basis for the topology, one can find a basic open subset W in $\mathcal{B}$ such that $x \in W \subset U$, and thus we have $x \in W \subset \overline{W} \subset \overline{U} \subset V$.■

$(\Longleftarrow)$ Suppose that X is $\mathbf{T_1}$ and satisfies the condition in the exercise. If U is an open subset and $x \in U$, let V be a basic open set from $\mathcal{B}$ such that $x \in V \subset U$, and let W be the basic open set that exists by the hypothesis in the exercise. Then we have $x \in W \subset \overline{W} \subset V \subset U$ and therefore X is regular.■

(b) Prove that the space constructed in Exercise VI.1.2 is $\mathbf{T_3}$. [*Hint:* Remember that the "new" topology contains the usual metric topology.]

SOLUTION.

By the first part of the exercise we only need to show this for points in basic open subsets. If the basic open subset comes from the metric topology, this follows because the metric topology is $\mathbf{T_3}$; note that the closure in the new topology might be smaller than the closure in the metric topology, but if a metrically open set contains the metric closure it also contains the "new" closure. If the basic open subset has the form $T_\varepsilon(a)$ for some a and z belongs to this set, there are two cases depending upon whether or not z lies on the x-axis, which we again call A. If $z \notin A$, then z lies in the metrically open subset $T_\varepsilon(a) - A$, and one gets a subneighborhood whose closure lies in the latter exactly as before. On the other hand, if $z \in A$ then $z = (a, 0)$ and the closure of the set $T_\delta(a)$ in either topology is contained in the set $T_\varepsilon(a)$.■

3. If X is a topological space and $A \subset X$ is nonempty then X/A (in words, "X mod A" or "X modulo A collapsed to a point") is the quotient space whose equivalence classes are A and all one point subsets $\{x\}$ such that $x \notin A$. Geometrically, one is collapsing A to a single point.

(a) Suppose that A is closed in X. Prove that X/A is Hausdorff if either X is compact Hausdorff or X is metric (in fact, if X is $\mathbf{T_3}$).

SOLUTION.

If X is $\mathbf{T_3}$, then for each point $x \notin A$ there are disjoint open subsets U and V such that $x \in U$ and $A \subset V$. Let $\pi : X \to X/A$ be the quotient projection; we claim that $\pi[U]$ and $\pi[V]$ are open disjoint subsets of X/A. Disjointness follows immediately from the definition of the equivalence relation, and the sets are open because their inverse images are the open sets $U = \pi^{-1}[\,\pi[U]\,]$ and $V = \pi^{-1}[\,\pi[V]\,)$ respectively.■

(b) Still assuming A is closed but not making any assumptions on X (except that it be nonempty), show that the quotient map $X \to X/A$ is always closed but not necessarily open. [*Note:* For reasons that we shall not discuss, it is appropriate to define $X/\emptyset$ to be the disjoint union $X \sqcup \{\emptyset\}$.]

SOLUTION.

Suppose that $F \subset X$ is closed; we need to show that $\pi^{-1}[\,\pi(F)\,]$ is also closed. There are two cases depending upon whether or not $A \cap F = \emptyset$. If the two sets are disjoint, then $\pi^{-1}[\,\pi(F)\,] = F$, and if the intersection is nonempty then $\pi^{-1}[\,\pi(F)\,] = F \cup A$. In either case the inverse image is closed, and therefore the image of F is always closed in the quotient space.

We claim that π is not necessarily open if A has a nonempty interior. Suppose that both of the statements in the previous sentence are true for a specific example, and let v be a nonempty open subset of X that is contained in A. If π is open then $\pi[V] = \pi[A] = \{A\} \in X/A$ is an open set. Therefore its inverse image, which is A, must be open in X. But it is also closed in X. Therefore we have the following conclusion: *If A is a nonempty proper closed subset of the connected space X with a nonempty interior, then $\pi : X \to X/A$ is not an open mapping.*■

(c) Suppose that we are given a continuous map of topological spaces $f : X \to Y$, and that $A \subset X$ and $B \subset Y$ are nonempty closed subsets satisfying $f[A] \subset B$. Prove that there is a unique continuous map $F : X/A \to Y/B$ such that for all $\mathbf{c} \in X/A$, if $\mathbf{c}$ is the equivalence class of $x \in X$, then $F(\mathbf{c})$ is the equivalence class of $f(x)$.

SOLUTION.

Let $p : X \to X/A$ and $q : Y \to Y/B$ be the projection maps, and consider the composite $q \circ f$. Then the condition $f[A] \subset B$ implies that $q \circ f$ sends each equivalence class for the relation defining X/A to a point in Y/B, and thus by the basic properties of quotient maps it follows that there is a unique continuous map $F : X/A \to Y/B$ such that $q \circ f = F \circ p$; this is equivalent to the conclusion stated in this part of the problem.■

VI.4 : Local compactness and compactifications

Problems from Munkres, § 38, pp. 241 – 242

2. Show that the bounded continuous function $g : (0,1) \to [-1,1]$ defined by $g(x) = \cos(1/x)$ cannot be extended continuously to the compactification in Example 3 [on page 238 of Munkres]. Define an embedding $h : (0,1) \to [-1,1]^3$ such that the functions x, $\sin(1/x)$ and $\cos(1/x)$ are all extendible to the compactifcation given by the closure of the image of h.

SOLUTION.

We need to begin by describing the compactification mentioned in the exercise. It is given by taking the closure of the embedding (homeomorphism onto its image) $g : (0,1) \to [-1,1]^2$ that is inclusion on the first coordinate and $\sin(1/x)$ on the second.

Why is it impossible to extend $\cos(1/x)$ to the closure of the image? Look at the points in the image with coordinates $(1/k\pi, \sin k\pi)$ where k is a positive integer. The second coordinates of these points are always 0, so this sequence converges to the origin. If there is a continuous extension F, it will follow that

$$F(0,0) \quad = \quad \lim_{k\to\infty} \cos\left(\frac{1}{1/k\pi}\right) \quad = \quad \cos k\pi \ .$$

But the terms on the right hand side are equal to $(-1)^k$ and therefore do not have a limit as $k \to \infty$. Therefore no continuous extension to the compactification exists.

One can construct a compactification on which x, $\sin(1/x)$ and $\cos(1/x)$ extend by taking the closure of the image of the embedding $h : (0,1) \to [-1,1]^3$ defined by

$$h(x) \quad = \quad (x, \sin(1/x), \cos(1/x)\,) \ .$$

The continuous extensions are given by restricting the projections onto the first, second and third coordinates.■

3. [*Just give a necessary condition on the topology of the space.*] Under what conditions does a metrizable space have a metrizable compactification?

SOLUTION.

If A is a dense subset of a compact metric space, then A must be second countable because a compact metric space is second countable and a subspace of a second countable space is also second countable.

This condition is also sufficient, but the sufficiency part was not assigned because it requires the Urysohn Metrization Theorem. The latter says that a $\mathbf{T_3}$ and second countable topological space is homeomorphic to a product of a countably infinite product of copies of $[0,1]$; this space is compact by Tychonoff's Theorem, and another basic result not covered in the course states that a countable product of metrizable spaces is metrizable in the product topology (see Munkres, Exercise 3 on pages 133–134). So if X is metrizable and second countable, the Urysohn Theorem maps it homeomorphically to a subspace of a compact metrizable space, and the closure of its image will be a metrizable compactification of X.■

Additional exercises

Definition. If $f : X \to Y$ is continuous, then f is *proper* (or *perfect*) if for each compact subset $K \subset Y$ the inverse image $f^{-1}[K]$ is a compact subset of X.

1. Suppose that $f : X \to Y$ is a continuous map of noncompact locally compact $\mathbf{T_2}$ spaces. Let $f^\bullet : X^\bullet \to Y^\bullet$ be the map of one point comapactifications defined by $f^\bullet | X = f$ and $f^\bullet(\infty_X) = (\infty_Y)$. Prove that f is proper if and only if $f^\bullet$ is continuous.

SOLUTION.

$(\Longrightarrow)$ Suppose that f is proper and U is open in $Y^\bullet$. There are two cases depending upon whether $\infty_Y \in U$. If not, then $U \subset Y$ and thus $[f^\bullet]^{-1}[U] = f^{-1}[U]$ is an open subset of X; since X is open in $X^\bullet$ it follows that $f^{-1}[U]$ is open in $X^\bullet$. On the other hand, if $\infty_Y \in U$ then $Y - U$ is compact, and since f is proper it follows that

$$C \quad = \quad f^{-1}[Y - U] \quad = \quad X - f^{-1}[U - \{\infty_Y\}]$$

is a compact, hence closed, subset of X and $X^\bullet$. Therefore

$$[f^\bullet]^{-1}[U] \quad = \quad f^{-1}[U] \cup \{\infty_X\} \quad = \quad X^\bullet - C$$

is an open subset of $X^\bullet$.■

$(\Longleftarrow)$ Suppose that $f^\bullet$ is continuous and $A \subset Y$ is compact. Then

$$f^{-1}[A] = (f^\bullet)^{-1}[A]$$

is a closed, hence compact subset of $X^\bullet$ and likewise it is a compact subset of X.■

2. Prove that a proper map of noncompact locally compact Hausdorff spaces is closed.

SOLUTION.

Let $f : X \to Y$ be a proper map of noncompact locally compact Hausdorff spaces, and let $f^\bullet$ be its continuous extension to a map of one point compactifications. Since the latter are compact Hausdorff it follows that $f^\bullet$ is closed. Suppose now that $F \subset X$ is closed. If F is compact, then so is $f[F]$ and hence the latter is closed in Y. Suppose now that F is not compact, and consider the closure E of F in $X^\bullet$. This set is either F itself or $F \cup \{\infty_X\}$ (since F is its own closure in X it follows that $E \cap X = F$). Since the closed subset $E \subset X^\bullet$ is compact, clearly $E \neq F$, so this implies the second alternative. Once again we can use the fact that $f^\bullet$ is closed to show that $f^\bullet[E] = f[F] \cup \{\infty_Y\}$ is closed in $Y^\bullet$. But the latter equation implies that $f[F] = f^\bullet[F] \cap Y$ is closed in Y.■

3. If $\mathbf{F}$ is the reals or complex numbers, prove that every polynomial map $p : \mathbf{F} \to \mathbf{F}$ is proper. [Hint: Show that

$$\lim_{|z| \to \infty} \bigl|p(z)\bigr| = \infty$$

and use the characterization of compact subsets as closed and bounded subsets of $\mathbf{F}$.]

SOLUTION.

Write the polynomial as

$$p(z) \quad = \quad \sum_{k=0}^{n} a_k \, z^k$$

where $a_n \neq 0$ and $n > 0$, and rewrite it in the following form:

$$a_n z^n \cdot \left(1 \quad + \quad \sum_{k=1}^{n-1} \frac{a_k}{a_n} \frac{1}{z^{n-k}} \right)$$

The expression inside the parentheses goes to 1 as $n \to \infty$, so we can find $N_0 > 0$ such that $|z| \geq N_0$ implies that the abolute value (or modulus) of this expression is at least $\frac{1}{2}$.

Let $M > 0$ be arbitrary, and define

$$N_1 \quad = \quad \left(\frac{2M_0}{|a_n|}\right)^{1/n} \quad .$$

Then $|z| > \max(N_0,\, N_1)$ implies $|p(z)| > M$. This proves the limit formula.

To see that p is proper, suppose that K is a compact subset of $\mathbf{F}$, and choose $M > 0$ such that $w \in K$ implies $|w| \leq M$. Let N be the maximum of N_1 and N_2, where these are defined as in the preceding paragraph. We then have that the closed set $p^{-1}[K]$ lies in the bounded set of points satisfying $|z| \leq N$, and therefore $p^{-1}[K]$ is a compact subset of $\mathbf{F}$.■

4. Let ℓ^2 be the complete metric space described above, and view $\mathbb{R}^n$ as the subspace of all sequences with $x_k = 0$ for $k > n$. Let $A_n \subset \ell^2 \times \mathbb{R}$ be the set of all ordered pairs (x,t) with $x \in \mathbb{R}^n$ and $0 < t \leq 2^{-n}$. Show that $A = \bigcup_n A_n$ is locally compact but its closure is not. Explain why this shows that the completion of a locally compact metric space is not necessarily locally compact. [*Hint:* The family $\{\, A_n \,\}$ is a locally finite family of closed locally compact subspaces in A. Use this to show that the union is locally compact, and show that the closure of A contains all of $\ell^2 \times \{0\}$. Explain why ℓ^2 is not locally compact.]

SOLUTION.

As usual, we follow the hints.

> [*Show that*] *the family* $\{\, A_n \,\}$ *is a locally finite family of closed locally compact subspaces in* A.

If $(x,t) \in \ell^2 \times (0, +\infty)$ and U is the open set $\ell^2 \times (t/2,\, +\infty)$, then $x \in U$ and $A_n \cap U = \emptyset$ unless $2^{-n} \geq t/2$, and therefore the family is locally finite. Furthermore, each set is closed in $\ell^2 \times (0, +\infty)$ and therefore also closed in $A = \bigcup_n A_n$.

> *Use this to show that the union is locally compact.*

We shall show that *if* A *is a* $\mathbf{T_3}$ *space that is a union of a locally finite family of closed locally compact subsets* A_α*, then* A *is locally compact.* Let $x \in A$, and let U be an open subset of A containing x such that $U \cap A_\alpha = \emptyset$ unless $\alpha = \alpha_1 \;\cdots\;, \alpha_k$. Let V_0 be an open subset of A such that $x \in V_0 \subset \overline{V_0} \subset U$, for each i choose an open set $W_i \in A_{\alpha_i}$ such that the closure of W_i is compact, express W_i as an intersection $V_i \cap A_{\alpha_i}$ where V_i is open in A, and finally let $V = \cap_i V_i$. Then we have

$$\overline{V} \quad = \quad \bigcup_{i=1}^{k} \left(\overline{V} \cap A_{\alpha_i}\right) \quad \subset \quad \bigcup_{i=1}^{k} \left(\overline{V_i} \cap A_{\alpha_i}\right) \quad = \quad \bigcup_{i=1}^{k} \textbf{Closure}\,(W_i \;\text{ in }\; A_{\alpha_i}) \; .$$

Since the set on the right hand side is compact, the same is true for $\overline{V}$. Therefore we have shown that A is locally compact.

> *Show that the closure of* A *contains all of* $\ell^2 \times \{0\}$*. Explain why* ℓ^2 *is not locally compact.*

Let $x \in \ell^2$, and for each positive integer k let $P_k(x) \in A_k$ be the point $(H_k(x), 2^{-(k+1)}\,)$, where $H_k(x)$ is the point whose first k coordinates are those of x and whose remaining coordinates are 0. It is an elementary exercise to verify that $(x,0) = \lim_{k\to\infty} P_k(x)$. To conlclude we need to show that ℓ^2 is not locally compact. If it were, then there would be some $\varepsilon > 0$ such that the set of all $y \in \ell^2$ satisfying every $|y| \leq \varepsilon$ would be compact, and consequently infinite sequence $\{\, y_n \,\}$ in ℓ^2

with $|y_n| \leq \varepsilon$ (for all n) would have a convergent subsequence. To see this does not happen, let $y_k = \frac{1}{2}\varepsilon\,\mathbf{e}_k$, where $\mathbf{e}_k$ is the k^{th} standard unit vector in ℓ^2. This sequence satifies the boundedness condition but does not have a convergent subsequence. Therefore ℓ^2 is not locally compact.■

FOOTNOTE.

Basic theorems from functional analysis imply that a normed vector space is locally compact if and only if it is finite-dimensional.■

5. Let X be a compact Hausdorff space, and let $U \subset X$ be open and noncompact. Prove that the collapsing map $c : X \to U^\bullet$ such that $c|U = \mathrm{id}_U$ and $c = \infty_U$ on $X - U$ is continuous. Show also that c is not necessarily open.

SOLUTION.

Suppose that V is open in $U^\bullet$. If $\infty_U \notin V$ then $V \subset U$ and $c^{-1}[V] = V$, so the set on the left hand side of the equation is open. Suppose now that $\infty_U \in V$; then $A = U^\bullet - V$ is a compact subset of U and $c^{-1}[V] = X - c^{-1}[A] = X - A$, which is open because the compact set A is also closed in X.

There are many examples for which c is not open. For example, let $X = [0, 5]$ and $A = [1, 3]$; in this example the image J of the open set $(2, 4)$ is not open because the inverse image of J is $[1, 4)$, which is not open. More generally, if X is connected and $X - U$ has a nonempty interior, then $X \to U^\bullet$ is not open (try to prove this!).■

FOOTNOTE.

In fact, if $F = X - U$ then $U^\bullet$ is homeomorphic to the space X/F described in a previous exercise. This is true because the collapse map passes to a continuous map from X/F to $U^\bullet$ that is 1–1 onto, and this map is a homeomorphism because X/F is compact and $U^\bullet$ is Hausdorff.■

6. **(a)** Explain why a compact Hausdorff space has no nontrivial Hausdorff abstract closures.

SOLUTION.

If X is compact Hausdorff and $f : X \to Y$ is a continuous map into a Hausdorff space, then $f[X]$ is closed. Therefore $f[X] = Y$ if the image of f is dense, and in fact f is a homeomorphism.■

(b) Prove that a Hausdorff space X has a maximal abstract Hausdorff closure that is unique up to equivalence. [*Hint:* Consider the identity map.]

SOLUTION.

This exercise shows that a formal analog of an important concept (the Stone-Čech compactification) is not necessarily as useful as the original concept itself; of course, there are also many situations in mathematics where the exact opposite happens. In any case, given an abstract closure (Y, f) we must have $(X, \mathrm{id}_X) \geq (Y, f)$ because $f : X \to Y$ trivially satisfies the condition $f = f \circ \mathrm{id}_X$.■

7. Suppose that X is compact Hausdorff and A is a closed subset of X. Prove that X/A is homeomorphic to the one point compactification of $X - A$.

SOLUTION.

See the footnote to Exercise 5 above.■

8. Suppose that X is a metric space that is uniformly locally compact in the sense that there is some $\delta > 0$ such that for each $x \in X$ the neighborhood $N_\delta(x)$ has compact closure. Prove that X is complete. Explain why the conclusion fails if one only assumes that for each x there is some $\delta(x) > 0$ with the given property (give an example).

SOLUTION.

Suppose that X is uniformly locally compact as above and $\{x_n\}$ is a Cauchy sequence in X. Choose M such that $m, n \geq M$ implies $\mathbf{d}(x_m, x_n) < \delta$, where δ is as in the problem. If we define a new sequence by $y_k = x_{k-M}$, then $\{y_k\}$ is a Cauchy sequence whose values lie in the (compact) closure of $N_\delta(x_M)$, and since compact metric spaces are complete the sequence $\{y_k\}$ has some limit z. Since $\{y_k\}$ is just $\{x_{k-M}\}$, it follows that the sequence $\{x_{k-M}\}$, and hence also the sequence $\{x_n\}$, must converge to the same limit z.

One of the simplest examples of a noncomplete metric space with the weaker property is the half-open interval $(0, \infty)$. In this case, if we are given x we can take $\delta_x = x/2$.■

9. Suppose that X is a noncompact locally compact $\mathbf{T}_2$ space and A is a noncompact closed subset of X. Prove that the one point compactification $A^\bullet$ is homeomorphic to a subset of the one point compactification $X^\bullet$.

SOLUTION.

Let $B = A \cup \{\infty_X\}$, and consider the function $g : A^\bullet \to X$ such that g is the inclusion on A and g sends ∞_A to ∞_X. By construction this map is continuous at all ordinary points of A, and if g is also continuous at ∞_A then g will define a 1–1 continuous map from $A^\bullet$ onto B, and this map will be a homeomorphism.

Suppose now that V is an open neighborhood of ∞_X in $X^\bullet$. By definition we know that $K = X - V$ is compact. But now we have

$$g^{-1}[V] \;\; = \;\; A^\bullet \; - \; g^{-1}[X - V] \;\; = \;\; A^\bullet \; - \; A \cap K \; .$$

Now $A \cap K$ is compact because both A and $X - V$ are closed X and K is a compact subset of the Hausdorff space X, and therefore it follows that the set $g^{-1}[V] \; = \; A^\bullet \; - \; A \cap K$ is open in A, so that g is continuous everywhere.■

10. Let U be an open subset of $\mathbb{C}$, let $a \in U$, and let $f : U - \{a\} \to \mathbb{C}$ be a continuous function with the following properties: (i) There is an open neighborhood $V \subset U$ of a such that f is nonzero on $V - \{a\}$. (ii) We have

$$\lim_{z \to a} \frac{1}{f(z)} \;\; = \;\; 0 \; .$$

If $F : U \to \mathbb{C}^\bullet$ is defined by $F|U - \{a\} = f$ and $F(a) = \infty$, prove that F is continuous. [*Hint:* It is only necessary to show that F is continuous at a, which means that for each compact subset $K \subset \mathbb{C}$ there is some $\delta > 0$ such that $0 < |z - a| < \delta$ implies that $f(z) \notin K$.]

SOLUTION.

Follow the hint. The final sentence is equivalent to the desired conclusion because an arbitrary open neighborhood of ∞ has the form $\{\infty\} \, \cup \, (\mathbb{C} - K)$, where K is a compact subset of $\mathbb{C}$. Since the compact set K is bounded in $\mathbb{C}$ we can find some $M > 0$ such that all points of K have distance strictly less than M from the origin. Let $\varepsilon = 1/M$, and using the limit assumption for $1/f$ at a let δ be chosen so that $0 < |z - a| < \delta$ implies $|1/f(z)| < \varepsilon$. Then the given conditions on z also imply $|f(z)| > M$, so that $f(z) \notin K$, which is what we needed to show.■

VI.5 : Metrization theorems

Problems from Munkres, § 40, p. 252

2. A subset W of X is said to be an $\mathbf{F}_\sigma$ set in X if W is a countable union of closet subsets of X. Show that W is an $\mathbf{F}_\sigma$ set in X if and only if $X - W$ is a $\mathbf{G}_\delta$ set in X. [The reason for the terminology is discussed immediately following this exercise on page 252 of Munkres.]

SOLUTION.

If W is an $\mathbf{F}_\sigma$ set then $W = \cup_n \, F_n$ where n ranges over the nonnegative integers and F_n is closed in X. Therefore

$$X - W \quad = \quad \bigcap_n X - F_n$$

is a countable intersection of the open subsets $X - F_n$ and accordingly is a $\mathbf{G}_\delta$ set.

Conversely, if $V = X - W$ is a $\mathbf{G}_\delta$ set, then $V = \cap_n \, U_n$ where n ranges over the nonnegative integers and U_n is open in X. Therefore

$$W \quad = \quad X - V \quad = \quad \bigcup_n X - U_n$$

is a countable union of the closed subsets $X - U_n$ and accordingly is an $\mathbf{F}_\sigma$ set.■

FOOTNOTE.

We have already shown that a closed subset of a metrizable space is a $\mathbf{G}_\delta$ set, and it follows that every open subset of a metrizable space is an $\mathbf{F}_\sigma$ set.■

3. Many spaces have *countable* bases, but no $\mathbf{T_1}$ space has a *locally finite* basis unless it is discrete. Prove this fact.

SOLUTION.

Suppose that X is $\mathbf{T_1}$ and has a locally finite base $\mathcal{B}$. Then for each $x \in X$ there is an open set W_x containing x such that $W_x \cap V_\beta = \emptyset$ for all β except $\beta(1), \; \cdots \; , \beta(k)$. Let V^* be the intersection of all sets in the finite subcollection that contain x. Since $\mathcal{B}$ is a base for this topology it follows that some $V_{\beta(J)}$ contains x and is contained in this intersection, But this means that $V_{\beta(J)}$ must be contained in all the other open subsets in $\mathcal{B}$ that contain x and is therefore a *minimal* open subset containing x. Suppose this minimal open set has more than one point; let y be another point in the set. Since X is $\mathbf{T_1}$ it will follow that $V_{\beta(J)} - \{y\}$ is also an open subset containing x. However, this contradicts the minimality of $V_{\beta(J)}$ and shows that the latter consists only of the point $\{x\}$. Since x was arbitrary, this shows that every one point subset of X is open and thus that X is discrete.■

Additional exercises

1. A *pseudometric space* is a pair $(X, \mathbf{d})$ consisting of a nonempty set X and a function $\mathbf{d} X \times X \to \mathbb{R}$ that has all the properties of a metric except possibly the property that $\mathbf{d}(u, v) = 0$.

(a) If ε-neighborhoods and open sets are defined as for metric spaces, explain why one still obtains a topology for pseudometric spaces.

SOLUTION.

None of the arguments verifying the axioms for open sets in metric spaces rely on the assumption $\mathbf{d}(x,y) = 0 \Longrightarrow x = y$.■

(b) Given a pseudometric space, define a binary relation $x \sim y$ if and only if $\mathbf{d}(x,y) = 0$. Show that this defines an equivalence relation and that $\mathbf{d}(x,y)$ only depends upon the equivalence classes of x and y.

SOLUTION.

The relation is clearly reflexive and symmetric. To see that it is transitive, note that $\mathbf{d}(x,y) = \mathbf{d}(y,z) = 0$ and the Triangle Inequality imply

$$\mathbf{d}(x,z) \quad \leq \quad \mathbf{d}(x,y) \; + \; \mathbf{d}(y,z) \quad = \quad 0 \; + \; 0 \quad = \quad 0 \; .$$

Likewise, if $x \sim x'$ and $y \sim y'$ then

$$\mathbf{d}(x',y') \quad \leq \quad \mathbf{d}(x',x) \; + \; \mathbf{d}(x,y) \; + \; \mathbf{d}(y,y') \quad = \quad 0 \; + \; \mathbf{d}(x,y) \; + \; 0 \quad = \quad \mathbf{d}(x,y)$$

and therefore the distance between two points only depends upon their equivalence classes with respect to the given relation.■

(c) Given a sequence of pseudometrics $\mathbf{d}_n$ on a set X, let $\mathbf{T}_\infty$ be the topology generated by the union of the sequence of topologies associated to these pseudometrics, and suppose that for each pair of distinct points $u\, v \in X$ there is some n such that $\mathbf{d}_n(u,v) > 0$. Prove that $(X, \mathbf{T}_\infty)$ is metrizable and that

$$\mathbf{d}_\infty \quad = \quad \sum_{n=1}^{\infty} \frac{\mathbf{d}_n}{2^n(1+\mathbf{d}_n)}$$

defines a metric whose underlying topology is $\mathbf{T}_\infty$.

SOLUTION.

First of all we verify that $\mathbf{d}_\infty$ defines a metric. In order to do this we must use some basic properties of the function

$$\varphi(x) \quad = \quad \frac{x}{1+x} \; .$$

This is a continuous and strictly increasing function defined on $[0, +\infty)$ and taking values in $[0,1)$ and it has the additional property $\varphi(x+y) \leq \varphi(x) + \varphi(y)$. The continuity and monotonicity properties of φ follow immediately from a computation of its derivative, the statement about its image follows because $x \geq 0$ implies $0 \leq \varphi(x) < 1$ and $\lim_{x\to+\infty} \varphi(x) = 1$ (both calculations are elementary exercises that are left to the reader).

The inequality $\varphi(x+y) \leq \varphi(x) + \varphi(y)$ is established by direct computation of the difference $\varphi(x) + \varphi(y) - \varphi(x+y)$:

$$\frac{x}{1+x} \; + \; \frac{y}{1+y} \; - \; \frac{x+y}{1+x+y} \quad = \quad \frac{x^2y \; + \; 2\,xy \; + \; xy^2}{(1+x)\,(1+y)\,(1+x+y)}$$

This expression is nonnegative if x and y are nonnegative, and therefore one has the desired inequality for φ. Another elementary but useful inequality is $\varphi(x) \leq x$ if $x \geq 0$ (this is true because $1 \leq 1 + x$). Finally, we note that the inverse to the continuous strictly monotonic function φ is given by

$$\varphi^{-1}(y) \quad = \quad \frac{y}{1-y} \; .$$

It follows that if $\mathbf{d}$ is a pseudometric then so is $\varphi \circ \mathbf{d}$ with the additional property that $\varphi \circ \mathbf{d} \leq 1$. More generally, if $\{ a_n \}$ is a convergent sequence of nonnegative real numbers and $\{ \mathbf{d}_n \}$ is a sequence of pseudometrics on a set X, then

$$\mathbf{d}_\infty \quad = \quad \sum_{n=1}^{\infty} a_n \cdot \varphi \circ \mathbf{d}_n < \sum_{n=1}^{\infty} a_n < \infty$$

also defines a pseudometric on X (write out the details of this!). In our situation $a_n = 2^{-n}$. Therefore the only thing left to prove about $\mathbf{d}_\infty$ is that it is positive when $x \neq y$. But in our situation if $x \neq y$ then there is some n such that $\mathbf{d}_n(x,y) > 0$, and the latter in turn implies that

$$2^{-n}\varphi\big(\mathbf{d}_n(x,y) \big) \quad > \quad 0$$

and since the latter is one summand in the infinite sum of nonnegative real numbers given by $\mathbf{d}_\infty(x,y)$ it follows that the latter is also positive. *Therefore $\mathbf{d}_\infty$ defines a metric on X.*

To prove that the topology $\mathcal{M}$ defined by this metric and the topology $\mathbf{T}_\infty$ determined by the sequence of pseudometrics are the same. Let N_α denote an α-neighborhood with respect to the $\mathbf{d}_\infty$ metric, and for each n let $N_\beta^{\langle n \rangle}$ denote a β-neighborhood with respect to the pseudometric $\mathbf{d}_n$. Suppose that N_ε is a basic open subset for $\mathcal{M}$ where $\varepsilon > 0$ and $x \in X$. Choose A so large that $n \geq A$ implies

$$\sum_{k=A}^{\infty} 2^{-k} \quad < \quad \frac{\varepsilon}{2}$$

Let W_x be the set of all z such that $\mathbf{d}_k(x,z) < \varepsilon/2$ for $1 \leq k < A$. Then W_x is the finite intersection of the $\mathbf{T}_\infty$-open subsets

$$W^{\langle k \rangle}(x) \quad = \quad \{ \ z \in X \ \mid \ \mathbf{d}_k(x,z) < \varepsilon/2 \ \}$$

and therefore W_k is also $\mathbf{T}_\infty$-open. Direct computation shows that if $y \in W_k$ then

$$\mathbf{d}_\infty(x,y) \quad = \quad \sum_{n=1}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \quad = \quad \sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \ + \ \sum_{n=A}^{\infty} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \quad <$$

$$\left(\sum_{n=1}^{A-1} 2^{-n}\,\varphi\,(\,\mathbf{d}_n(x,y)\,) \right) \ + \ \frac{\varepsilon}{2} \quad < \quad \left(\sum_{n=1}^{A-1} \frac{2^{-n}\,\varepsilon}{2} \right) \ + \ \frac{\varepsilon}{2} \quad < \quad \frac{\varepsilon}{2} \ + \ \frac{\varepsilon}{2} \quad < \quad \varepsilon$$

so that $W_x \subset N_\varepsilon(x)$. Therefore, if U is open with respect to $\mathbf{d}_\infty$ we have

$$U \quad = \quad \bigcup_{x \in U} \{x\} \quad \subset \quad \bigcup_{x \in U} W_x \quad \subset \quad \bigcup_{x \in U} N_\varepsilon(x) \quad \subset \quad U$$

which shows that U is a union of $\mathbf{T}_\infty$-open subsets and therefore is $\mathbf{T}_\infty$-open. *Thus $\mathcal{M}$ is contained in $\mathbf{T}_\infty$.*

To show the reverse inclusion, consider an subbasic $\mathbf{T}_\infty$-open subset of the form $N_\varepsilon^{\langle n \rangle}(x)$. If y belongs to the latter then there is a $\delta > 0$ such that $N_\delta^{\langle n \rangle}(y) \subset N_\varepsilon^{\langle n \rangle}(x)$; without loss of generality we may as well assume that $\delta < 2^{-k}$. If we set

$$\eta(y) \quad = \quad 2^{-k}\,\varphi\big(\,\delta(y)\,\big)$$

then $\mathbf{d}_\infty(z,y) < \eta(y)$ and

$$2^{-k}\,\varphi\circ\mathbf{d}_k \quad\le\quad \mathbf{d}_\infty$$

imply that $2^{-k}\,\varphi\circ\mathbf{d}_k(z,y) < \eta(y)$ so that $\varphi\circ\mathbf{d}_k(z,y) < 2^k\,\eta(y)$ and

$$\mathbf{d}_k(z,y) \quad<\quad \varphi^{-1}\big(2^k\,\delta(y)\big)$$

and by the definition of $\eta(y)$ the right hand side of this equation is equal to $\delta(y)$. Therefore if we set $W = N^{\langle k\rangle}_\varepsilon(x)$ then we have

$$W \quad=\quad N^{\langle k\rangle}_\varepsilon(x) \quad=\quad \bigcup_{y\in W}\{y\} \quad\subset\quad \bigcup_{y\in W} N_{\eta(y)}(y) \quad\subset\quad \bigcup_{y\in W} N^{\langle k\rangle}_{\delta(y)}(y) \quad\subset\quad N^{\langle k\rangle}_\varepsilon(x)$$

which shows that $N^{\langle k\rangle}_\varepsilon(x)$ belongs to $\mathcal{M}$. *Therefore the topologies* $\mathbf{T}_\infty$ *and* $\mathcal{M}$ *are equal.*■

(d) Let X be the set of all continuous real valued functions on the real line $\mathbb{R}$. Prove that X is metrizable such that the restriction maps from X to $\mathbf{BC}([-n,n])$ are uniformly continuous for all n. [*Hint:* Let $\mathbf{d}_n(f,g)$ be the maximum value of $|f(x)-g(x)|$ for $|x|\le n$.]

SOLUTION.

Take the pseudometrics $\mathbf{d}_n$ as in the hint, and given $h\in X$ let $J^{\langle n\rangle}(h)$ be its restriction to $[-n,n]$. Furthermore, let $\|...\|_n$ be the uniform metric on $\mathbf{BC}([-n,n])$, so that

$$\mathbf{d}_n(f,g) \quad=\quad \|\,J^{\langle n\rangle}(f) \;-\; J^{\langle n\rangle}(g)\,\|_n\ .$$

Given $\varepsilon > 0$ such that $\varepsilon < 1$, let $\delta = 2^{-n}\,\varphi(\varepsilon)$. Then $\mathbf{d}_\infty(f,g) < \delta$ implies $2^{-n}\,\varphi\circ\mathbf{d}_n(f,g) < \delta$, which in turn implies $\mathbf{d}_n(f,g) < \varphi^{-1}(2^n\,\delta) = \varepsilon$.■

(e) Given X and the metric constructed in the previous part of the problem, prove that a sequence of functions $\{f_n\}$ converges to f if and only if for each compact subset $K\subset\mathbb{R}$ the sequence of restricted functions $\{f_n|K\}$ converges to $f|K$.

SOLUTION.

We first claim that $\lim_{n\to\infty}\mathbf{d}_\infty(f_n,f) = 0$ if and only if $\lim_{n\to\infty}\mathbf{d}_k(f_n,f) = 0$ for all k.

$(\implies)$ Let $\varepsilon > 0$ and fix k. Since $\lim_{n\to\infty}\mathbf{d}_\infty(f_n,f) = 0$ there is a positive integer M such that $n\ge M$ implies $\mathbf{d}_\infty(f_n,f) < 2^k\,\varphi(\varepsilon)$. Since

$$2^{-k}\varphi\circ\mathbf{d}_k \quad\le\quad \mathbf{d}_\infty$$

it follows that $\mathbf{d}_k \le \varphi^{-1}(2^{-k}\,\mathbf{d}_\infty)$ and hence that $\mathbf{d}_k(f_n,f) < \varepsilon$ if $n\ge M$.■

$(\impliedby)$ Let $\varepsilon > 0$ and choose A such that

$$\sum_{k=n}^{\infty} 2^{-k} \quad<\quad \frac{\varepsilon}{2}\ .$$

Now choose B so that $n\ge B$ implies that

$$\mathbf{d}_k(f_n,f) \quad<\quad \frac{\varepsilon}{2A}$$

for all $k < A$. Then if $n \geq A + B$ we have

$$\mathbf{d}_\infty(f_n, f) \;=\; \sum_{k=1}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;=\; \sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \sum_{k=A}^{\infty} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;<$$

$$\sum_{k=1}^{A} 2^{-k}\,\varphi^\circ\mathbf{d}_k(f_n, f) \;+\; \frac{\varepsilon}{2} \;<\; \left(\sum_{k=1}^{A} 2^{-k}\frac{\varepsilon}{2A}\right) \;+\; \frac{\varepsilon}{2} \;=\; \varepsilon$$

so that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$.■

Now suppose that K is a compact subset of $\mathbb{R}$, and let $\|...\|_K$ be the uniform norm on $\mathbf{BC}(K)$. Then $K \subset [-n, n]$ for some n and thus for all $g \in X$ we have $\|g\|_K \leq \mathbf{d}_n(g, 0)$. Therefore if $\{ f_n \}$ converges to f then the sequence of restricted functions $\{ f_n|K \}$ converges uniformly to $f|K$. Conversely, if for each compact subset $K \subset \mathbb{R}$ the sequence of restricted functions $\{ f_n|K \}$ converges to $f|K$, then this is true in particular for $K = [-L, L]$ and accordingly $\lim_{n\to\infty} \mathbf{d}_L(f_n, f) = 0$ for all L. However, as noted above this implies that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ and hence that $\{ f_n \}$ converges to f.■

(f) Is X complete with respect to the metric described above? Prove this or give a counterexample.

SOLUTION.

The answer is **YES**, and here is a proof: Let $\{ f_n \}$ be a Cauchy sequence in X. Then if $K_m = [-m, m]$ the sequence of restricted functions $\{ f_n|K_m \}$ is a Cauchy sequence in $\mathbf{BC}(K_m)$ and therefore converges to a limit function $g_m \in \mathbf{BC}(K_m)$. Since $\lim_{n\to\infty} f_n|K_m(x) = g_m(x)$ for all $x \in K_m$ it follows that $p \leq m$ implies $g_m|K_p = g_p$ for all such m and p. Therefore if we define $g(x) = g_m(x)$ if $|x| < m$ then the definition does not depend upon the choice of m and the continuity of g_m for each m implies the continuity of g. Furthermore, by construction it follows that

$$\lim_{n\to\infty} \mathbf{d}_m(f_n, g) \;=\; \lim_{n\to\infty} \|\,(f_n|K_m) \;-\; g_m\,\| \;=\; 0$$

for all m and hence that $\lim_{n\to\infty} \mathbf{d}_\infty(f_n, f) = 0$ by part **(e)** above.■

(g) Explain how the preceding can be generalized from continuous functions on $\mathbb{R}$ to continuous functions on an arbitrary open subset $U \subset \mathbb{R}^n$.

SOLUTION.

The key idea is to express U as an increasing union of bounded open subsets V_n such that $\overline{V_n} \subset V_{n+1}$ for all n. If U is a proper open subset of $\mathbb{R}^n$ let $F = \mathbb{R}^n - U$ (hence F is closed), and let V_m be the set of all points x such that $|x| < m$ and $\mathbf{d}_{[\mathbf{2}]}(x, F_m) > 1/m$, where $\mathbf{d}_{[\mathbf{2}]}$ denotes the usual Euclidean metric; if $U = \mathbb{R}^n$ let V_m be the set of all points x such that $|x| < m$. Since $y \in U$ if and only if $\mathbf{d}_{[\mathbf{2}]}(y, F) = 0$, it follows that $U = \cup_m V_m$. Furthermore, since $y \in \overline{V_m}$ implies $|x| \leq m$ and $\mathbf{d}_{[\mathbf{2}]}(x, F_n) \geq 1/n$ (why?), we have $\overline{V_m} \subset V_{m+1}$. Since $\overline{V_m}$ is bounded it is compact.

If f and g are continuous real valued functions on U, define $\mathbf{d}_n(f, g)$ to be the maximum value of $|f(x) - g(x)|$ on $\overline{V_m}$. In this setting the conclusions of parts **(d)** through **(f)** go through with only one significant modification; namely, one needs to check that every compact subset of U is contained in some V_m. To see this, note that K is a compact subset that is disjoint from the closed subset F, and therefore the continuous function $\mathbf{d}_{[\mathbf{2}]}(y, F)$ assumes a positive minimum value c_1 on K and that there is a positive constant c_2 such that $y \in K$ implies $|y| \leq c_2$. If we choose m such that $m > 1/c_1$ and $m > c_2$, then K will be contained in V_m as required.■

FOOTNOTE.

Here is another situation where one encounters metrics defined by an infinite sequence of pseudometrics. Let Y be the set of all infinitely differentiable functions on $[0,1]$, let D^k denote the operation of taking the k^{th} derivative, and let $\mathbf{d}_k(f,g)$ be the maximum value of $|D^k f \ - \ D^k g|$, where $0 \le k < \infty$. One can also mix this sort of example with the one studied in the exercise; for instance, one can consider the topology on the set of infinitely differentiable functions on $\mathbb{R}$ defined by the countable family of pseudometrics $\mathbf{d}_{k,n}$ where $\mathbf{d}_{k,n}(f,g)$ is the maximum of $|D^k f \ - \ D^k g|$ on the closed interval $[-n,n]$.■

2. **(a)** Let U be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be a continuous function such that $f^{-1}(\{0\})$ is contained in an open subset V such that $V \subset \overline{V} \subset U$. Prove that there is a continuous function g from $S^n \cong (\mathbb{R}^n)^\bullet$ to itself such that $g|V = f|V$ and $f^{-1}[\{0\}] = g^{-1}[\{0\}]$. [*Hint:* Note that

$$(\mathbb{R}^n)^\bullet \ - \ \{0\} \quad \cong \quad \mathbb{R}^n$$

and consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \quad \subset \quad (\mathbb{R}^n)^\bullet \ - \ \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbb{R}^n)^\bullet \ - \ V$ with the same codomain? What happens if we try to piece this together with the original function f defined on U?]

SOLUTION.

We shall follow the steps indicated in the hint(s).

Note that

$$(\mathbb{R}^n)^\bullet \ - \ \{0\} \quad \cong \quad \mathbb{R}^n$$

This is true because the left hand side is homeomorphic to the complement of a point in S^n, and such a complement is homeomorphic to $\mathbb{R}^n$ via stereographic projection (which may be taken with respect to an arbitrary unit vector on the sphere).■

Consider the continuous function on

$$\left(\overline{V} - V\right) \sqcup \{\infty\} \quad \subset \quad (\mathbb{R}^n)^\bullet \ - \ \{0\}$$

defined on the respective pieces by the restriction of f and ∞. Why can this be extended to a continuous function on $(\mathbb{R}^n)^\bullet \ - \ V$ with the same codomain?

In the first step we noted that the codomain was homeomorphic to $\mathbb{R}^n$, and the Tietze Extension Theorem implies that a continuous function from a closed subset A of a metric space X into $\mathbb{R}^n$ extends to all of X.■

What happens if we try to piece this together with the original function f defined on U?

If h is the function defined above, then we can piece h and $f|\overline{V}$ together and obtain a continuous function on all of $(\mathbb{R}^n)^\bullet$ if and only if the given functions agree on the intersection of the two closed subsets. This intersection is equal to $\overline{V} - V$, and by construction the restriction of h to this subset is equal to the restriction of f to this subset.■

(b) Suppose that we are given two continuous functions g and g' satisfying the conditions of the first part of this exercise. Prove that there is a continuous funtion

$$G : S^n \times [0,1] \longrightarrow S^n$$

such that $G(x, 0) = g(x)$ for all $x \in S^n$ and $G(x, 1) = g'(x)$ for all $x \in S^n$ (*i.e.*, the mappings g and g' are **homotopic**).

SOLUTION.

Let h and h' denote the associated maps from $(\mathbb{R}^n)^\bullet - V$ to $(\mathbb{R}^n)^\bullet - \{0\} \cong \mathbb{R}^n$ that are essentially given by the restrictions of g and g'. Each of these maps has the same restriction to $\left(\overline{V} - V\right) \sqcup \{\infty\}$. Define a continuous mapping $H : ((\mathbb{R}^n)^\bullet - V) \times [0, 1] \longrightarrow (\mathbb{R}^n)^\bullet - \{0\} \cong \mathbb{R}^n$ by $H(x, t) = t\, h'(x) + (1 - t)\, h(x)$, and define F' on $\overline{V} \times [0, 1]$ by $F(x, t) = f(x)$. As in the first part of this exercise, the mappings H and F' agree on the intersection of their domains and therefore they define a continuous map G on all of $S^n \times [0, 1]$. Verification that $G(x, 0) = g(x)$ and $G(x, 1) = g'(x)$ is an elementary exercise.■

3. Suppose that X is a a metrizable space. Define G_δ and F_σ sets as in Section 40 of Munkres and its exercises.

(a) Show that every open subset of X is an F_σ set. [*Hint:* Look at Example 2 on page 249.]

SOLUTION.

Let U be open in X, and let $F = X - U$. Then by the examples cited above we know that F is a closed set which is also a G_δ set. Apply Exercise 2 on page 252 to show that $U = X - F$ must be an open F_σ set.■

(b) Show that a subset A of X is a closed G_δ set if and only if there is some continuous real valued function $f : X \to [0, 1]$ such that $A = f^{-1}[\,\{0\}\,]$.

SOLUTION.

If A is the zero set of a continuous function as above, then

$$A \quad = \quad \cap_n \geq 1\ f^{-1}\big[\,[0, 1/n)\,\big]$$

where each half open interval $[0, n)$ is open in $[0, 1]$, so that their inverse images are open in X. Conversely, if A is a closed G_δ of the form $\cap_n\, U_n$, then if we set $V_n = U_1 \cap\ \cdots\ \cap U_n$, we have $A = \cap_n\, V_n$ and $V_1 \supset V_2 \supset\ \cdots$. Define functgions $g_n : X \to [0, 1]$ such that $g_n = 0$ on A and $g_n = 1$ on $F_n = X - V_n$. If we take $f = \sum_n 2^{-n} \cdot g_n$, then the infinite series converges absolutely and uniformly to a continuous function, and the zero set of this function is precisely A.■

(c) Show that a subset W of X is an open F_σ set if and only if there is some continuous real valued function $f : X \to [0, 1]$ such that W is the set of all points x such that $f(x)$ is positive.

SOLUTION.

Let $A = X - W$, and let f be as in the previous part of this exercise. Then the set of zero points is A, so the set of points where f is nonzero (and in fact positive by construction) will be the set $W = X - A$.■

4. Let X be compact, and let $\mathcal{F}$ be a family of continuous real valued functions on X that is closed under multiplication and such that for each $x \in X$ there is a neighborhood U of X and a function $f \in \mathcal{F}$ that vanishes identically on U. Prove that $\mathcal{F}$ contains the zero function.

SOLUTION.

For each x let φ_x be a continuous real valued function in $\mathcal{F}$ such that $\varphi_x = 0$ on some open set U_x containing x. The family of open sets $\{\,U_x\,\}$ is an open covering of X and therefore has a finite subcovering by sets $U_{x(i)}$ for $1 \leq i \leq k$. The product of the functions $\prod_i f_{x(i)}$ belongs to $\mathcal{F}$ and is zero on each $U_{x(i)}$; since the latter sets cover X it follows that the product is the zero function and therefore the latter belongs to $\mathcal{F}$.■

5. Let X be a compact metric space, and let J be a nonempty subset of the ring $\mathbf{BC}(X)$ of (bounded) continuous functions on X such that J is closed under addition and subtraction, it is an *ideal* in the sense that $f \in J$ and $g \in \mathbf{BC}(X) \Longrightarrow f \cdot g \in J$, and for each $x \in X$ there is a function $f \in J$ such that $f(x) \neq 0$. Prove that $J = \mathbf{BC}(X)$. [*Hints:* This requires the existence of partitions of unity as established in Theorem 36.1 on pages 225–226 of Munkres; as noted there, the result works for arbitrary compact Hausdorff spaces, but we restrict to metric spaces because the course does not cover Urysohn's Lemma in that generality. Construct a finite open covering of X, say $\mathcal{U}$, such that for each $U_i \in \mathcal{U}$ there is a function $f_i \in J$ such that $f_i > 0$ on U_i. Let $\{\varphi_i\}$ be a partition of unity dominated by $\mathcal{U}$, and form $h = \sum_i \varphi_i \cdot f_i$. Note that $h \in J$ and $h > 0$ everywhere so that h has a reciprocal $k = 1/h$ in $\mathbf{BC}(X)$. Why does this imply that the constant function 1 lies in J, and why does the latter imply that everything lies in J?]

REMINDER.

If X is a topological space and $\mathcal{U} = \{U_1, \cdots, U_n\}$ is a finite indexed open covering of X, then a partition of unity subordinate to $\mathcal{U}$ is an indexed family of continuous functions $\varphi_i : X \to [0,1]$ for $1 \leq i \leq n$ such that for each i the zero set of the function φ_i contains contains $X - U_i$ in its interior and

$$\sum_{i=1}^{n} \varphi_i \quad = \quad 1 \ .$$

Theorem 36.1 on pages 225–226 of Munkres states that for each finite indexed open covering $\mathcal{U}$ of a $\mathbf{T_4}$ space (hence for each such covering of a compact Hausdorff space), there is a partition of unity subordinate to $\mathcal{U}$. The proof of this is based upon Urysohn's Lemma, so the methods in Munkres can be combined with our proof of the result for metric space to prove the existence of partitions of unity for indexed finite open coverings of compact metric spaces.

SOLUTION.

For each $x \in X$ we are assuming the existence of a continuous bounded function f_x such that $f_x(x) \neq 0$. Since J is closed under multiplication, we may replace this function by it square if necessary to obtain a function $g_x \in J$ such that $g_x(x) > 0$. Let U_x be the open set where g_x is nonzero, and choose an open subset V_x such that $\overline{V_x} \subset U_x$. The sets V_x determine an open covering of X; this open covering has a finite subcovering that we index and write as $\mathcal{V} = \{V_1, \cdots, V_n\}$. For each i let $g_i \in J$ be the previously chosen function that is positive on $\overline{V_i}$, and consider the function

$$g \quad = \quad \sum_{i=0}^{k} \varphi_i \cdot g_i \ .$$

This function belongs to J, and we claim that $g(y) > 0$ for all $y \in X$. Since $\sum_i \varphi_i = 1$ there is some index value m such that $\varphi_m(y) > 0$; by definition of a partition of unity this means that $y \in V_m$. But $y \in V_m$ implies $g_m(y) > 0$ too and therefore we have $g(y) \geq \varphi_m(y)\, g_m(y) > 0$. Since the reciprocal of a nowhere vanishing continuous real valued function on a compact space is continuous (and bounded!), we know that $1/g$ lies in $\mathbf{BC}(X)$. Since $g \in J$ it follows that $1 = g \cdot (1/g)$ also lies in J, and this in turn implies that $h = h \cdot 1$ lies in J for all $h \in \mathbf{BC}(X)$. Therefore $J = \mathbf{BC}(X)$ as claimed.■

6. In the notation of the preceding exercise, an ideal $\mathbf{M}$ in $\mathbf{BC}(X)$ is said to be a *maximal ideal* if it is a proper ideal and there are no ideals $\mathbf{A}$ such that $\mathbf{M}$ is properly contained in $\mathbf{A}$ and $\mathbf{A}$ is properly contained in in $\mathbf{BC}(X)$. Prove that there is a 1–1 correspondence between the maximal ideals of $\mathbf{BC}(X)$ and the points of X such that the ideal $\mathbf{M}_x$ corresponding to X is the set of all continuous functions $g : X \to \mathbb{R}$ such that $g(x) = 0$. [*Hint:* Use the preceding exercise.]

SOLUTION.

Let $\mathcal{M}$ be the set of all maximal ideals. For each point $x \in X$ we need to show that $\mathbf{M}_x \in \mathcal{M}$. First of all, verification that $\mathbf{M}_x$ is an ideal is a sequence of elementary computations (which the reader should verify). To see that the ideal is maximal, consider the function $\widehat{x} : \mathbf{BC}(X) \to \mathbb{R}$ by the formula $\widehat{x}(f) = f(x)$. This mapping is a ring homomorphism, it is onto, and $\widehat{x}(f) = 0$ if and only if $f \in \mathbf{M}_x$. Suppose that the ideal is not maximal, and let $\mathbf{A}$ be an ideal such that $\mathbf{M}_x$ is properly contained in $\mathbf{A}$. Let $a \in A$ be an element that is not in $\mathbf{M}_x$. Then $a(x) = \alpha \neq 0$ and it follows that $a(x) - \alpha 1$ lies in $\mathbf{M}_x$. It follows that $\alpha 1 \in \mathbf{A}$, and since $\mathbf{A}$ is an ideal we also have that $1 = \alpha^{-1}(\alpha 1)$ lies in $\mathbf{A}$; the latter in turn implies that every element $f = f \cdot 1$ of $\mathbf{BC}(X)$ lies in $\mathbf{A}$.

We claim that the map from X to $\mathcal{M}$ sending x to $\mathbf{M}_x$ is 1–1 and onto. Given distinct points x and y there is a bounded continuous function f such that $f(x) = 0$ and $f(y) = 1$, and therefore it follows that $\mathbf{M}_x \neq \mathbf{M}_y$. To see that the map is onto, let $\mathbf{M}$ be a maximal ideal, and note that the preceding exercise implies the existence of some point $p \in X$ such that $f(p) = 0$ for all $f \in \mathbf{M}$. This immediately implies that $\mathbf{M} \subset \mathbf{M}_p$, and since both are maximal (proper) ideals it follows that they must be equal. Therefore the map from X to $\mathcal{M}$ is a 1–1 correspondence.■

FOOTNOTES.

(1) One can use techniques from functional analysis and Tychonoff's Theorem to put a natural topology on $\mathcal{M}$ (depending only on the structure of $\mathbf{BC}(X)$ as a Banach space and an algebra over the reals) such that the correspondence above is a homeomorphism; see page 283 of Rudin, *Functional Analysis*, for more information about this..

(2) The preceding results are the first steps in the proof of an important result due to I. Gelfand and M. Naimark that give a complete set of abstract conditions under which a Banach algebra is isomorphic to the algebra of continuous complex valued functions on a compact Hausdorff space. A Banach algebra is a combination of Banach space and associative algebra (over the real or complex numbers) such that the multiplication and norm satisfy the compatibility relation $|xy| \leq |x| \cdot |y|$. The additional conditions required to prove that a Banach algebra over the complex numbers is isomorphic to the complex version of $\mathbf{BC}(X)$ are commutativity, the existence of a unit element, and the existence of an conjugation-like map (formally, an **involution**) $a \to a^*$ satisfying the additional condition $|aa^*| = |a|^2$. Details appear in Rudin's book on functional analysis, and a reference for the Gelfand-Naimark Theorem is Theorem 11.18 on page 289. A classic reference on Banach algebras (definitely NOT up-to-date in terms of current knowledge but an excellent source for the material it covers) is the book by Rickart in the bibliographic section of the course notes.

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 6

Fall 2008

APPENDICES

Appendix A : Topological groups

(Munkres, Supplementary exercises following $ 22; *see also* course notes, Appendix D)

Problems from Munkres, § 30*, pp.* 194 – 195

Munkres, § 26, pp. 170–172: 12, 13

Munkres, § 30, pp. 194–195: 18

Munkres, § 31, pp. 199–200: 8

Munkres, § 33, pp. 212–214: 10

Solutions for these problems will not be given.

Additional exercises

Notation. Let $\mathbb{F}$ be the real or complex numbers. Within the matrix group $\mathbf{GL}(n, \mathbb{F})$ there are certain subgroups of particular importance. One such subgroup is the *special linear group* $\mathbf{SL}(n, \mathbb{F})$ of all matrices of determinant 1.

0. Prove that the group $\mathbf{SL}(2, \mathbb{C})$ has no nontrivial proper normal subgroups except for the subgroup $\{\pm I\}$. [*Hint:* If N is a normal subgroup, show first that if $A \in N$ then N contains all matrices that are similar to A. Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here α is a complex number not equal to 0 or ± 1 and $\varepsilon = \pm 1$. The idea is to show that if N contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.]

SOLUTION.

We shall follow the steps indicated in the hint.

If N is a normal subgroup of $\mathbf{SL}(2, \mathbb{C})$ and $A \in N$ then N contains all matrices that are similar to A.

If B is similar to A then $B = P\,A\,P^{-1}$ for some $P \in \mathbf{GL}(2,\mathbb{C})$, so the point is to show that one can choose P so that $\det P = 1$. The easiest way to do this is to use a matrix of the form βP for some nonzero complex number β, and if we choose the latter so that $\beta^2 = (\det P)^{-1}$ then βP will be a matrix in in $\mathbf{SL}(2,\mathbb{C})$ and B will be equal to $(\beta P)\,A\,(\beta P)^{-1}$.■

Therefore the proof reduces to considering normal subgroups containing a Jordan form matrix of one of the following two types:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} \varepsilon & 1 \\ 0 & \varepsilon \end{pmatrix}$$

Here α is a complex number not equal to 0 or ± 1 and $\varepsilon = \pm 1$.

The preceding step that if N contains a given matrix then it contains all matrices similar to that matrix. Therefore if N is a nontrivial normal subgroup of $\mathbf{SL}(2,\mathbb{C})$ then N is a union of similarity classes, and the latter in turn implies that N contains a given matrix A if and only if it contains a Jordan form for A. For 2×2 matrices there are only two basic Jordan forms; namely, diagonal matrices and elementary 2×2 Jordan matrices of the form

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

where $\lambda \neq 0$. Such matrices have determinant 1 if and only if the product of the diagonal entries is the identity, and this means that the Jordan forms must satisfy conditions very close in the hint; the only difference involves the diagonal case where the diagonal entries may be equal to ± 1. If N is neither trivial nor equal to the subgroup $\{\pm I\}$, then it must contain a Jordan form given by a diagonal matrix whose nonzero entries are not ± 1 or else it must contain an elementary 2×2 Jordan matrix.

The idea is to show that if N contains one of these Jordan forms then it contains all such forms, and this is done by computing sufficiently many matrix products. Trial and error is a good way to approach this aspect of the problem.

The basic strategy is to show first that if N contains either of the matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \quad , \quad \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

(where $\alpha \neq 0,\ \pm 1$), then N also contains

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and next to show that if N contains the latter matrix then N contains every Jordan form. As noted before, the latter will imply that $N = \mathbf{SL}(2,\mathbb{C})$.

At this point one needs to do some explicit computations to find products of matrices in N with sufficiently many Jordan forms. Suppose first that N contains the matrix

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

Then N contains A^2, which is equal to

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and the latter is similar to

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

so N contains B. Suppose now that N contains some diagonal matrix A whose nonzero entries are α and α^{-1} where $\alpha \neq \pm 1$, and let B be given as above. Then N also contains the commutator $C = A\,B\,A^{-1}\,B^{-1}$. Direct computation shows that the latter matrix is given as follows:

$$C = \begin{pmatrix} 1 & \alpha^2 - 1 \\ 0 & 1 \end{pmatrix}$$

Since $\alpha \neq \pm 1$ it follows that C is similar to B and therefore $B \in N$.

We have now shown that if N is a nontrivial normal subgroup that is not equal to $\{\pm I\}$, then N must contain B. As noted before, the final step is to prove that if $B \in N$ then $N = \mathbf{SL}(2, \mathbb{C})$.

Since $B \in N$ implies that N contains all matrices similar to B it follows that all matrices of the forms

$$P = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \quad , \qquad Q = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$$

(where z is an arbitrary complex number)

also belong to N, which in turn shows that $P\,Q \in N$. But.

$$P\,Q = \begin{pmatrix} 1 & z \\ z & z^2 + 1 \end{pmatrix}$$

and its characteristic polynomial is $t^2 - (z^2 + 2)\,t - 1$. If $z \neq 0$ then this polynomial has distinct nonzero roots such that one is the reciprocal of the other. Furthermore, if $\alpha \neq 0$ then one can always solve the equation $\alpha + \alpha^{-1} = z^2 + 2$ for z over the complex numbers and therefore we see that for each $\alpha \neq 0$ there is a matrix $P\,Q$ similar to

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

such that $P\,Q \in N$. It follows that each of the displayed diagonal matrices also lies in N; in particular, this includes the case where $\alpha = -1$ so that the diagonal matrix is equal to $-I$. To complete the argument we need to show that N also contains a matrix similar to

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} .$$

But this is relatively easy because N must contain $-B = (-I)B$ and the latter is similar to A.■

FOOTNOTE.

More generally, if $\mathbb{F}$ is an arbitrary field then every proper normal subgroup of $\mathbf{SL}(n, \mathbb{F})$ is contained in the central subgroup of diagonal matrices of determinant 1; of course, the latter is isomorphic to the group of n^{th} roots of unity in $\mathbb{F}$.

Definition. The *orthogonal group* $\mathbf{O}(n)$ consists of all transformations in $\mathbf{GL}(n, \mathbb{R})$ that take each orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis, or equivalently, the subgroup of all matrices whose columns form an orthonormal basis. It is an easy exercise in linear algebra to show that the determinant of all matrices in $\mathbf{O}(n)$ is ± 1. The *special orthogonal group* $\mathbf{SO}(n)$ is the subgroup of $\mathbf{O}(n)$ consisting of all matrices whose determinants are equal to $+1$. Replacing If we replace the real numbers $\mathbb{R}$ by the complex numbers $\mathbb{C}$ we get the unitary groups $\mathbf{U}(n)$ and the *special unitary groups* $\mathbf{SU}(n)$, which are the subgroups of $\mathbf{U}(n)$ given by matrices with determinant 1. The determinant of every matrix in $\mathbf{U}(n)$ is of absolute value 1 just as before, but in the complex case this means that the determinant is a complex number on the unit circle. In Appendix A the orthogonal and unitary groups were shown to be compact.

1. Show that $\mathbf{O}(1)$ is isomorphic to the cyclic group of order 2 and that $\mathbf{SO}(2)$ is isomorphic as a topological group to the circle group S^1. Conclude that $\mathbf{O}(2)$ is homeomorphic as a space to $\mathbf{SO}(2) \times \mathbf{O}(1)$, but that as a group these objects are not isomorphic to each other. [*Hint:* In Appendix D it is noted that every element of $\mathbf{O}(2)$ that is not in $\mathbf{SO}(2)$ has an orthonormal basis of eigenvectors corresponding to the eigenvalues ± 1. What does this say about the orders of such group elements?]

SOLUTION.

The group $\mathbf{GL}(n, \mathbb{R})$ is isomorphic to the multiplicative group of nonzero real numbers. Therefore $\mathbf{O}(1)$ is isomorphic to set of nonzero real numbers that take 1 to an element of absolute value 1; but a nonzero real number has this property if and only if its absolute value is 1, or equivalently if it is equal to ± 1.

Consider now the group $\mathbf{SO}(2)$. Since it sends the standard basis into an orthonormal basis, it follows that its columns must be orthonormal. Therefore there is some θ such that the entries of the first column are $\cos\theta$ and $\sin\theta$, and there is some φ such that the entries of the second column are $\cos\varphi$ and $\sin\varphi$. Since the vectors in question are perpendicular, it follows that $|\theta - \varphi| = \pi/2$, and depending upon whether the sign of $\theta - \varphi$ is positive or negative there are two possibilities:

(1) If $\theta - \phi > 0$ then $\cos\varphi = -\sin\theta$ and $\sin\varphi = \cos\theta$.

(2) If $\theta - \phi < 0$ then $\cos\varphi = \sin\theta$ and $\sin\varphi = -\cos\theta$.

In the first case the determinant is 1 and in the second it is -1.

We may now construct the isomorphism from S^1 to $\mathbf{SO}(2)$ as follows: If $z = x + y\,i$ is a complex number such that $|z| = 1$, send z to the matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} .$$

This map is clearly 1–1 because one can retrieve z directly from the entries of the matrix. It is onto because the complex number $\cos\theta + i\,\sin\theta$ maps to the matrix associated to θ. Verification that the map takes complex products to matrix products is an exercise in bookkeeping.

A homeomorphism from $\mathbf{SO}(2) \times \mathbf{O}(1)$ to $\mathbf{O}(2)$ may be constructed as follows: Let $\alpha : \mathbf{O}(1) \to \mathbf{O}(2)$ be the map that sends ± 1 to the diagonal matrix whose entries are ± 1 and 1, and define $M(A, \varepsilon)$ to be the matrix product $A\,\alpha(\varepsilon)$. Since the image of -1 does not lie in $\mathbf{SO}(2)$, standard results on cosets in group theory imply that the map M is 1–1 and onto. But it is also continuous, and since it maps a compact space to a Hausdorff space it is a homeomorphism onto its image, which is $\mathbf{SO}(2)$.

To see that this map is not a group isomorphism, note that in the direct product group $\mathbf{SO}(2) \times \mathbf{O}(1)$ there are only finitely many elements of order 2 (specifically, the first coordinate must be $\pm I$). On the other hand, the results from Appendix D show that all elements of $\mathbf{O}(2)$ that are not in $\mathbf{SO}(2)$ have order 2 and hence there are infinitely many such element in $\mathbf{O}(2)$. Therefore the latter cannot be isomorphic to the direct product group.■

2. Show that $\mathbf{U}(1)$ is isomorphic to the circle group S^1 as a topological group.

SOLUTION.

This is similar to the first part of the preceding exercise. The group $\mathbf{GL}(n, \mathbb{C})$ is isomorphic to the multiplicative group of nonzero complex numbers. Therefore $\mathbf{U}(1)$ is isomorphic to set of nonzero complex numbers that take 1 to an element of absolute value 1; but a nonzero complex number has this property if and only if its absolute value is 1, or equivalently if it lies on the circle S^1.■

3. For each positive integer n, show that $\mathbf{SO}(n)$, $\mathbf{U}(n)$ and $\mathbf{SU}(n)$ are connected spaces and that $\mathbf{U}(n)$ is homeomorphic as a space (but not necessarily isomorphic as a topological group) to $\mathbf{SU}(n) \times S^1$. [*Hints:* In the complex case use the Spectral Theorem for normal matrices to show that every unitary matrix lies in the path component of the identity, which is a normal subgroup. In the real case use the results on normal form in Appendix D.]

SOLUTION.

The proof separates into three cases depending upon whether the group in question is $\mathbf{U}(n)$, $\mathbf{SU}(n)$ or $\mathbf{SO}(n)$,

The case $G = \mathbf{U}(n)$.

We start with the unitary group. The Spectral Theorem states that if A is an $n \times n$ unitary matrix then there is another invertible unitary matrix P such that $B = P\,A\,P^{-1}$ is diagonal. We claim that there is a continuous curve joining B to the identity. To see this, write the diagonal entries of B as $\exp it_j$ where t_j is real and $1 \leq j \leq n$. Let $C(s)$ be the continuous curve in the diagonal unitary matrices such that the diagonal entries of $C(s)$ are $\exp(i\,st_j)$ where $s \in [0,1]$. It follows immediately that $C(0) = I$ and $C(1) = B$. Finally, if we let $\gamma(s) = P^{-1}\,C(s)\,P$ then γ is a continuous curve in $\mathbf{U}(n)$ such that $\gamma(0) = I$ and $\gamma(1) = A$. This shows that $\mathbf{U}(n)$ is in fact arcwise connected.■

The case $G = \mathbf{U}(n)$.

The preceding argument also shows that $\mathbf{SU}(n)$ is arcwise connected; it is only necessary to check that if A has determinant 1 then everything else in the construction also has this property. The determinants of B and A are equal because the determinants of similar matrices are equal. Furthermore, since the determinant of B is 1 it follows that $\sum_j \; t_j = 0$, and the latter in turn implies that the image of the curve $C(s)$ is contained in $\mathbf{SU}(n)$. Since the latter is a normal subgroup of $\mathbf{U}(n)$ it follows that the curve $\gamma(s)$ also lies in $\mathbf{SU}(n)$, and therefore we conclude that the latter group is also arcwise connected.

The product decomposition for $\mathbf{U}(n)$ is derived by an argument similar to the previous argument for $\mathbf{O}(2)$. More generally, suppose we have a group G with a normal subgroup K and a second subgroup H such that $G = H \cdot K$ and $H \cap K = \{1\}$. Then group theoretic considerations yield a 1–1 onto map $\varphi : K \times H \to G$ given by $\varphi(k,h) = k \cdot h$. If G is a topological group and the subgroups have the subspace topologies then the 1–1 onto map φ is continuous. Furthermore, if G is compact Hausdorff and H and K are closed subgroups of G then φ is a homeomorphism. In our particular situation we can take $G = \mathbf{U}(n)$, $K = \mathbf{SU}(n)$ and $H \cong \mathbf{U}(1)$ to be the subgroup of diagonal matrices with ones down the diagonal except perhaps for the $(1,1)$ entry. These subgroups satisfy all the conditions we have imposed and therefore we have that G is homeomorphic to the product $K \times H$.■

The case $G = \mathbf{SO}(n)$.

Finally, we need to verify that $\mathbf{SO}(n)$ is arcwise connected, and as indicated in the hint we use the normal form obtained in Appendix D. According to this result, for every orthogonal $n \times n$ matrix A there is another orthogonal matrix P such that $B = PAP^{-1}$ is is a block sum of orthogonal matrices that are either 2×2 or 1×1. These matrices may be sorted further using their determinants (recall that the determinant of an orthogonal matrix is ± 1).

In fact, one can choose the matrix P such that the block summands are sorted by size and determinant such that
(1) the 1×1 summands with determinant 1 come first,
(2) the 2×2 summands with determinant 1 come second,
(3) the 2×2 summands with determinant -1 come third,
(4) the 1×1 summands with determinant -1 come last.

Each matrix of the first type is a rotation matrix R_θ where the first row has entries $(\cos\theta \quad -\sin\theta)$, and each matrix of the second type is a matrix S_θ where the first row has entries $(\cos\theta \quad \sin\theta)$.

The first objective is to show that B lies in the same arc component as a matrix with no summands of the second type. Express the matrix B explicitly as a block sum as follows:

$$B = I_k \oplus \left(\bigoplus_{i=1}^{p} R_{\theta(i)} \right) \oplus \left(\bigoplus_{j=1}^{q} R_{\varphi(j)} \right) \oplus -I_\ell$$

Here I_m denotes an $m \times m$ identity matrix. Consider the continuous curve in $\mathbf{SO}(n)$ defined by the formula

$$C_1(t) = I_k \oplus \left(\bigoplus_{i=1}^{p} R_{t\theta(i)} \right) \oplus I_{2q+\ell} \ .$$

It follows that $C_1(1) = B$ and $C_1(0) = B_1$ is a block sum matrix with no summands of the second type.

The next objective is to show that B_1 lies in the same arc component as a matrix with no summands of the third type. This requires more work than the previous construction, and the initial step is to show that one can find a matrix in the same arc component with at most one summand of the third type. The crucial idea is to show that if the block sum has $q \geq 2$ summands of the third type, then it lies in the same arc component as a matrix with $q-2$ block summands of the third type. In fact, the continuous curve joining the two matrices will itself be a block sum, with one 4×4 summand corresponding to the two summands that are removed and identity matrices corresponding to the remaining summands. Therefore the argument reduces to looking at a 4×4 orthogonal matrix that is a block sum of two 2×2 matrices of the third type, and the objective is to show that such a matrix lies in the same arc component as the identity matrix.

Given real numbers α and β, let S_α and S_β be defined as above, and likewise for R_α and R_β. Then one has the multiplicative identity

$$S_\alpha \oplus S_\beta \quad = \quad \big(R_\alpha \oplus R_\beta\big) \cdot \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big) \ .$$

We have already shown that there is a continuous curve $C(t)$ in the orthogonal group such that $C(0) = I$ and $C(1) = R_\alpha \oplus R_\beta$. If we can construct a continuous curve $Q(t)$ such that $Q(0) = I$

and $Q(1) = \big(1 \oplus (-1) \oplus 1 \oplus (-1)\big)$. Then the matrix product $C(t)\,Q(t)$ will be a continuous curve in the orthogonal group whose value at 0 is the identity and whose value at 1 is $S_\alpha \oplus S_\beta$. But $R_\alpha \oplus R_\beta$ is a rotation by 180 degrees in the second and fourth coordinates and the identity on the first and third coordinates, so it is natural to look for a curve $Q(t)$ that is rotation through $180\,t$ degrees in the even coordinates and the identity on the odd ones. One can write down such a curve explicitly as follows:

$$Q(t) \quad = \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \pi t & 0 & -\sin \pi t \\ 0 & 0 & 1 & 0 \\ 0 & \sin \pi t & 0 & \cos \pi t \end{pmatrix}$$

Repeated use of this construction yields a matrix with no block summands of the first type and at most one summand of the second type. If there are no summands of the third type, we have reached the second objective, so assume that there is exactly one summand of the third type. Since our block sum has determinant 1, it follows that there is also at least one summand of the fourth type. We claim that the simplified matrix obtained thus far lies in the same arc component as a block sum matrix with no summands of the third type and one fewer summands of the fourth type. As in the previous step, everything reduces to showing that a 3×3 block sum $S_\alpha \oplus (-1)$ lies in the same arc component as the identity. In this case one has the multiplicative identity

$$S_\alpha \oplus (-1) \quad = \quad \big(R_\alpha \oplus 1\big) \oplus \Big(1 \oplus (-1) \oplus (-1)\Big)$$

and the required continuous curve is given by the formula

$$C(t) \quad = \quad \big(R_{t\alpha} \oplus 1\big) \oplus \Big((1 \oplus R_{t\pi}(-1)\Big) \ .$$

Thus far we have shown that every block sum matrix in $\mathbf{SO}(n)$ lies in the same arc component as a diagonal matrix (only summands of the first and last types). Let $I_s \oplus -I_\ell$ be this matrix. Since its determinant is 1, it follows that ℓ must be even, so write $\ell = 2k$. Then the diagonal matrix is a block sum of an identity matrix and k copies of R_π, and one can use the continuous curve $D(t)$ that is a block sum of an identity matrix with k copies of $R_{t\pi}$ to show that the diagonal matrix $I_s \oplus -I_\ell$ lies in the same arc component as the identity.

We have now shown that every block sum matrix with determinant 1 lies in the arc component of the identity. Suppose that $\Gamma(t)$ is a continuous curve whose value at 0 is the identity and whose value at 1 is the block sum. At the beginning of this proof we noted that for every orthogonal matrix A we can find another orthogonal matrix P such that $B = P\,A\,P^{-1}$ is is a block sum of the type discussed above; note that the determinant of B is equal to 1 if the same is true for the determinant of A. If $\Gamma(t)$ is the curve joining the identity to B, then $P^{-1}\,\Gamma(t)\,P$ will be a continuous curve in $\mathbf{SO}(n)$ joining the identity to A.■

4. Show that $\mathbf{O}(n)$ has two connected components both homeomorphic to $\mathbf{SO}(n)$ for every n.

SOLUTION.

One can use the same sort of arguments that established topological product decompositions for $\mathbf{O}(2)$ and $\mathbf{SU}(n)$ to show that $\mathbf{O}(n)$ is homeomorphic to $\mathbf{SO}(n) \times \mathbf{O}(1)$. Since $\mathbf{SO}(n)$ is connected, this proves that $\mathbf{O}(n)$ is homeomorphic to a disjoint union of two copies of $\mathbf{SO}(n)$.■

5. Show that the inclusions of $\mathbf{O}(n)$ in $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{U}(n)$ in $\mathbf{GL}(n, \mathbb{C})$ determine 1–1 correspondences of path components. [*Hint:* Show that the Gram-Schmidt orthonormalization process expresses an arbitrary invertible matrix over the real numbers as a product PQ where P is upper triangular and Q is orthogonal (real case) or unitary (complex case), and use this to show that every invertible matrix can be connected to an orthogonal or unitary matrix by a continuous curve.]

SOLUTION.

Let A be an invertible $n \times n$ matrix over the real or complex numbers, and consider the meaning of the Gram-Schmidt process for matrices. Let $\mathbf{a}_j$ represent the j^{th} column of A, and let B be the orthogonal or unitary matrix whose columns are the vectors $\mathbf{b}_j$ obtained from the columns of A by the Gram-Schmidt process. How are they related? The basic equations for defining the columns of B in terms of the columns of A have the form

$$\mathbf{b}_j = \sum_{k \leq j} c_{k,j} \mathbf{a}_k$$

where each $c_{j,j}$ is a positive real number, and if we set $c_{k,j} = 0$ for $k > j$ this means that $B = AC$ where C is lower triangular with diagonal entries that are positive and real. We claim that there is a continuous curve $\Gamma(t)$ such that $\Gamma(0) = I$ and $\Gamma(1) = C$. Specifically, define this curve by the following formula:

$$\Gamma(t) \quad = \quad I \ + \ t\,(C - I)$$

By construction $\Gamma(t)$ is a lower triangular matrix whose diagonal entries are positive real numbers, and therefore this matrix is invertible. If we let $\Phi(t)$ be the matrix product $A\,\Gamma(t)$ then we have a continuous curve in the group of invertible matrices such that $\Phi(1) = A$ and $\Phi(0)$ is orthogonal or unitary. Therefore it follows that every invertible matrix is in the same arc component as an orthogonal or unitary matrix.

In the unitary case this proves the result, for the arcwise connectedness of $\mathbf{U}(n)$ and the previous argument imply that $\mathbf{GL}(n, \mathbb{C})$ is also arcwise connected. However, in the orthogonal case a little more work is needed. The determinant of a matrix in $\mathbf{GL}(n, \mathbb{R})$ is either positive or negative, so the preceding argument shows that an invertible real matrix lies in the same arc component as the matrices of $\mathbf{SO}(n)$ if its determinant is positive and in the other arc component of $\mathbf{SO}(n)$ if its determinant is negative. Therefore there are at most two arc components of $\mathbf{GL}(n, \mathbb{R})$, and the assertion about arc components will be true if we can show that $\mathbf{GL}(n, \mathbb{R})$ is not connected. To see this, note that the determinant function defines a continuous onto map from $\mathbf{GL}(n, \mathbb{R})$ to the disconnected space $\mathbb{R} - \{0\}$.■