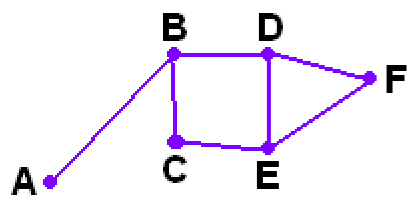
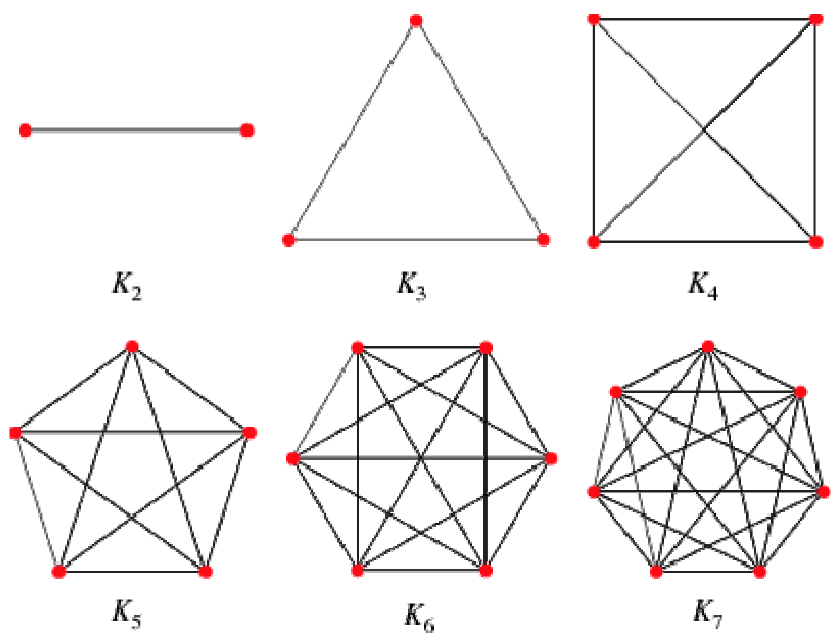
Examples of finite graph complexes

Many examples of finite graph complexes can be constructed very simply using pencil and paper. All that one must do is draw some dots on a piece of paper and join them by straight line segments such that no two of the segments meet unless they share a single end point. Here is a more or less random example:

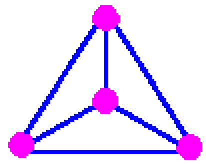


Notice that some but not all vertex pairs are the endpoints of an edge. If a graph has n vertices and every pair of vertices arises as the endpoints of some edge, then the graph is said to be the **complete graph** on the set of vertices. A complete graph on two vertices is just a closed line segment, and a complete graph on three vertices is just the triangle with the given set of vertices; it is a fairly straightforward exercise to prove that if X and Y are finite sets of vertices with the same number of points, then the complete graphs on X and Y are homeomorphic spaces. Here are examples of complete graphs on two to seven vertices:



(Source: http://mathworld.wolfram.com/CompleteGraph.html)

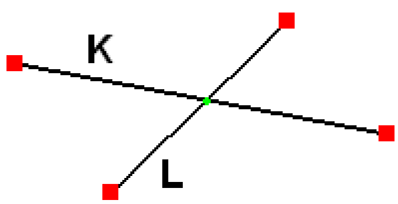
Strictly speaking, these pictures represent projections of the complete graphs onto a plane, so that two segments joining disjoint pairs of vertices do not have any points in common even though their planar projections do have a point in common. A complete graph on four vertices can be embedded nicely in the plane (see the picture below), but later in the course we shall show that a complete graph on five vertices is not homeomorphic to a subset of the coordinate plane; since this graph is contained in the complete graphs of six or more vertices, it also follows that none of these larger graphs can be realized. On the other hand, we shall also prove that every graph can be embedded rectilinearly (each edge is geometrically a closed segment) in coordinate $\mathbf{3}$ – space.

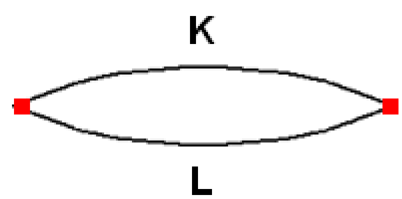


Example of a rectilinear embedding of the complete graph on four vertices into the plane

Note that every graph with n vertices can be obtained from a complete graph on n vertices by deleting some of the edges.

TWO NONEXAMPLES. Here are two unions of subsets **K** and **L**, each homeomorphic to a closed interval, such that their union does **not** define a graph. In the left hand example, the two edges meet at a point which is not an endpoint of either edge, while in the second the two edges meet in both endpoints. A third type of disallowed behavior is when the intersection of **K** and **L** is a vertex of one subset but not the other (think of **K** as the top of the letter **T** and **L** as the vertical stroke – drawing a picture of this will probably be helpful).





In each of these examples, it is possible to subdivide **K** and **L** into subintervals so that one does obtain a graph. Specifically, in the left hand example one can do this by splitting the two subsets at the point of intersection (the green point in the picture), while in the second one can do this by splitting each subset at its respective midpoint. Likewise, for the letter **T** example one splits one of the subsets at its midpoint.

Yet another type of disallowed behavior is if **K** and **L** have one endpoint in common and (say) the midpoint of **L** is the second endpoint of **K.** The reader should draw an example of this type (for example, look at the letter **P**) and show that one can also subdivide the edges in such cases to obtain a graph structure.