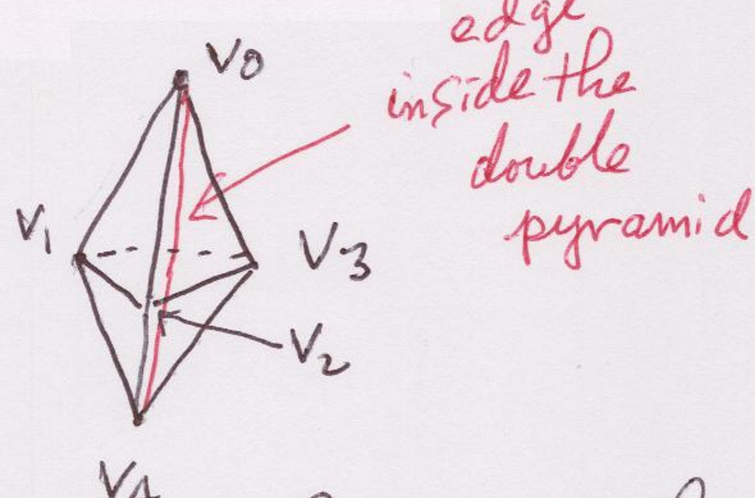
Example

Simplices are

$v_0 v_1 v_2$
$v_0 v_1 v_3$
$v_0 v_2 v_3$
$v_1 v_2 v_4$
$v_2 v_3 v_4$
$v_1 v_3 v_4$
$v_0 v_4$

and their faces.



Compute the homology.

Steps K_1 = upper pyramid faces.

$$K_1 = \overset{L_3}{v_0v_1v_2} \cup \overset{L_2}{v_0v_1v_3} \cup \overset{L_1}{v_0v_2v_3}.$$

Know H_* of each 2-simplex.

$L_1 \cap L_2$ = edge v_0v_3.

M-V sequence reduces to

$$\underset{H_q(L_1\cap L_2)}{\overset{\parallel}{0}} \longrightarrow \underset{H_q(L_1)\oplus H_q(L_2)}{\overset{\parallel}{0\oplus 0}} \longrightarrow H_q(L_1\cup L_2) \longrightarrow 0 \qquad \text{if } q \geq 2$$

so $H_q(L_1\cup L_2) = 0$ if $q \geq 2$.

Tail end

$$\begin{array}{ccc}
0\oplus 0 = H_1(L_1)\oplus H_1(L_2) \longrightarrow H_1(L_1\cup L_2) \xrightarrow{\Delta} & H_0(L_1\cap L_2) = \mathbb{Z} & 1 \\
& \downarrow \ \text{1-1 map} & \downarrow \\
& H_0(L_1)\oplus H_0(L_2) \quad \mathbb{Z}\oplus\mathbb{Z} & (1,-1) \\
& \downarrow \ \text{connected} & \\
& H_0(L_1\cup L_2) \quad \mathbb{Z} &
\end{array}$$

So $\Delta = 0$ and $H_1(L_1\cup L_2)$ is also trivial.

$K_1 = (L_1 \cup L_2) \cup L_3$, $(L_1 \cup L_2) \cap L_3 =$

$V_0 V_2 \cup V_0 V_1 \leftarrow$ homology zero if $q=0$, $\mathbb{Z}$ if $q=0$.

As before, can use MV to show

$$H_q(K_1) = \begin{cases} \mathbb{Z} & q=0 \\ 0 & q>0. \end{cases}$$

A similar conclusion holds for $K_2 =$ lower pyramid faces.

$K_1 \cap K_2 =$ triangle $V_1 V_2 V_3$.

Compute $H_*(K_1 \cup K_2)$ by M-V:

$$\underbrace{H_{q+1}(K_1) \oplus H_{q+1}(K_2)}_{0} \longrightarrow H_{q+1}(K_1 \cup K_2) \longrightarrow H_q(K_1 \cap K_2) \longrightarrow \underbrace{H_q(K_1) \oplus H_q(K_2)}_{0}$$

isomorphism if $q>0$.

So $H_r(K_1 \cup K_2) = \begin{cases} \mathbb{Z} & r=2 \\ 0 & \text{otherwise for } r \geq 3. \end{cases}$

Now look at the tail end.

$$\begin{matrix} H_1(K_1) \\ \oplus \\ H_1(K_2) \end{matrix} \longrightarrow H_1(K_1 \cup K_2) \longrightarrow H_0(K_1 \cap K_2) \longrightarrow \begin{matrix} H_0(K_1) \\ \oplus \\ H_0(K_2) \end{matrix}$$

$$0 \longrightarrow \qquad \xrightarrow{\Delta} \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$

$$1 \rightsquigarrow (1,-1)$$

so $\Delta = 0$ and
$H_1(K_1 \cup K_2) = 0.$

Let $M = K_1 \cup K_2$
$N = v_0 v_4.$

$K = M \cup N$
$M \cap N = \{v_0, v_4\}$?.

What do we get this time?

Need only worry about $q = 0, 1, 2$ since other chain groups $= 0$.

$$H_2(M\cap N) \to \begin{matrix} H_2(M) \\ \oplus \\ H_2(N) \end{matrix} \xrightarrow{\varphi} H_2(M\cup N) \longrightarrow$$

$$0 \longrightarrow \mathbb{Z}\oplus 0 \longrightarrow \boxed{\;?\;} \longrightarrow$$

$$H_1(M\cap N) \to \begin{matrix} H_1(M) \\ \oplus \\ H_1(N) \end{matrix} \longrightarrow H_1(M\cup N) \xrightarrow{\Delta}$$

$$0 \longrightarrow 0 \longrightarrow \boxed{\;?\;} \longrightarrow$$

$$H_0(M\cap N) \longrightarrow \begin{matrix} H_0(M) \\ \oplus \\ H_0(N) \end{matrix} \longrightarrow \begin{matrix} H_0(M\cup N) \\ = \mathbb{Z}. \end{matrix}$$

$$\mathbb{Z}\oplus\mathbb{Z} \xrightarrow{\theta} \mathbb{Z}\oplus\mathbb{Z} \xrightarrow[\text{COORDS}]{\text{ADD}} \mathbb{Z}$$

$$\begin{matrix} (1,0) \\ (0,1) \end{matrix} \rightsquigarrow (1,-1)$$

φ must be an isomorphism, so

$H_2(K) \cong \mathbb{Z}$. Kernel $\theta \cong \mathbb{Z}$ gen by $(1,-1)$.

Δ is 1-1 onto Kernel θ, so

$H_1(K) = \mathbb{Z}$ also.