Solution to Exercise 05.04

Given: $X \subseteq S^2$ such that $X = C_1 \cup C_2$ with $C_1, C_2 \cong S^1$ and $C_1 \cap C_2 = P = \{\text{point}\}$.

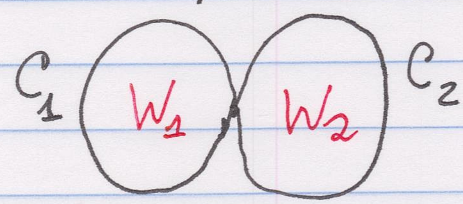
To prove: $S^2 - X$ has 3 components, where

(a) the boundary of one component is C_1,

(b) ——————" —————— C_2,

(c) the boundary of the third component is X.

Solution Here is a drawing of a standard example where $X \subseteq \mathbb{R}^2$:



Third component of $S^2 - X$ is unbounded.

We want to show that (a)–(c), which can be checked directly for the example, are true in complete generality.

The first step is to apply the Jordan Curve Thm (= 2D Jordan-Brouwer Separation thm.) to $S^2 - C_1$ and $S^2 - C_2$.

$$S^2 - C_1 = W_1 \cup V_1 \qquad S^2 - C_2 = W_2 \cup V_2$$

where W_i & V_i are (nonempty!) connected, $W_i \cap V_i = \emptyset$ and $\mathrm{Bdy}\, W_i = C_i = \mathrm{Bdy}\, V_i$.

Next, each set $C_i - P$ is connected and contained in $S^2 - C_j$, where $j \neq i$ $\left(\begin{matrix}\text{each is}\\ 1 \text{ or } 2\end{matrix}\right)$, so either $C_1 - P \subseteq W_2$ or $C_1 - P \subseteq V_2$

and $C_2 - P \subseteq W_1$ or $C_2 - P \subseteq V_1$.

Without loss of generality, we may relabel the sets V_i & W_i so that $C_i - P \subseteq V_i$.

(compare the drawing on the preceding page).

Hence $W_1, W_2 \subseteq S^2 - X$.

Now $W_i \subseteq S^2 - C_i$ is a maximal connected subset, and since $S^2 - X \subseteq S^2 - C_i$ it follows that

$$\boxed{\boxed{W_i \text{ is a component of } S^2 - X.}}$$

We shall now prove that $S^2 - X$ has exactly three components using the Mayer-Vietoris decomposition

$$S^2 - X = (S^2 - C_1) \cap (S^2 - C_2)$$
$$S^2 - P = (S^2 - C_1) \cup (S^2 - C_2).$$

(recall $X = C_1 \cup C_2$, $P = C_1 \cap C_2$).

The proof of the Jordan-Brouwer Thm. implies that $H_k(S^2 - C_i) = 0$ for $k > 0$, and since $S^2 - P \cong \mathbb{R}^2$ we have the following:

$$\begin{tikzcd}
H_1(S^2-P) \arrow[r, "\Delta"] \arrow[d, equal] & H_0(S^2-X) \arrow[r] \arrow[d, equal] & \begin{matrix} H_0(S^2-C_1) \\ \oplus \\ H_0(S^2-C_2) \end{matrix} \arrow[r, "\text{onto}"] \arrow[d, equal] & H_0(S^2-P) \arrow[d, equal] \\
0 \arrow[r] & ? \arrow[r, "\alpha"] & \begin{matrix} \mathbb{Z}^2 \\ \oplus \\ \mathbb{Z}^2 \end{matrix} \arrow[r, "\text{ADDITION}"] & \mathbb{Z}
\end{tikzcd}$$

It follows that α is 1-1 and its image is isomorphic to $\mathbb{Z}^3$. Hence $S^2 - X$ has 3 components. What are they?

W_1 and W_2 are distinct subsets because $\mathrm{Bdy}\, W_1 = C_1$ and $\mathrm{Bdy}\, W_2 = C_2$. Let Ω denote the remaining component of $S^2 - X$. To complete the proof, we need to show that $\mathrm{Bdy}(\Omega) = X$.

Enough to show $X \subseteq L(\Omega)$ [limit pts]

By construction, S^2 is the union of the pair-wise disjoint subsets

$$P,\ C_1 - P,\ C_2 - P,\ W_1,\ W_2,\ \Omega$$

where

$$V_1 = W_2 \cup (C_2 - P) \cup \Omega,$$
$$V_2 = W_1 \cup (C_1 - P) \cup \Omega.$$

Recall $S^2 - C_i = V_i \cup W_i$.

We first verify that $(C_1 - P) \subseteq L(\Omega)$. Let $z \in C_1 - P$. By Jordan-Brouwer, $z \in L(V_1)$.

Therefore, if N is an open neighborhood of z then we have $(N - \{z\}) \cap V_1 \neq \emptyset$. But $z \notin C_2$ and $z \notin L(W_2)$, so z does not lie in the closed set $F_2 = C_2 \cup W_2 = S^2 - V_2$. If $N' = N - F_2$, then N' is still an open neighborhood of z and

$$(N' - \{z\}) \cap V_1 \neq \emptyset.$$

Since $N' \cap F_2 = \emptyset$ by construction, this means that $(N' - \{z\}) \cap \Omega \neq \emptyset$ and hence $(N - \{z\}) \cap \Omega \neq \emptyset$, so that $z \in L(\Omega)$.

<u>Similarly</u>, $C_2 - P \subseteq L(\Omega)$, and since $L(\Omega)$ is closed it follows that $\overline{C_2 - P} = C_2 \subseteq L(\Omega)$. Combining this with the preceding discussion, we have $X = C_1 \cup C_2 \subseteq L(\Omega)$,

and hence $X \subseteq \mathrm{Bdy}\,\Omega = \overline{\Omega} - \Omega$.

Conversely, since $S^2 = \underbrace{W_1 \cup W_2 \cup \Omega \cup X}_{\text{PAIRWISE DISJOINT}}$,

the set $\Omega \cup X = S^2 - \underbrace{(W_1 \cup W_2)}_{\text{open}}$ is closed, so that

$$\mathrm{Bdy}(\Omega) = \overline{\Omega} - \Omega \subseteq (\Omega \cup X) - \Omega = X,$$

and hence $\mathrm{Bdy}\,(\Omega) = X$, which is what we wanted to prove. ∎