Comments on the Seifert – van Kampen Theorem

If we are given a topological space X which is the union of two open subsets U and V such that their intersection is arcwise connected and a base point $p \in U \cap V$, the Seifert-van Kampen Theorem describes $\pi_1(X,p)$ in terms of $\pi_1(U,p)$, $\pi_1(V,p)$ $\pi_1(U \cap V,p)$ and the maps of the latter into the first two groups induced by the inclusions $U \cap V \subset U$, $U \cap V \subset V$. Specifically, the images of $\pi_1(U)$ and $\pi_1(V,p)$ in $\pi_1(X,p)$ generate this group, and the kernel of the associated homomorphism from the free product $\pi_1(U,p) * \pi_1(V,p)$ to $\pi_1(X,p)$ is the normal subgroup which is normally generated by all elements of the form

$$i_{0*}(y) \cdot \big(i_{0*}(y) \big)^{-1}$$

where $i_0 : U \cap V \to V$ and $j_0 : U \cap V \to U$ are the inclusion mappings. One can characterize this situation in terms of group homomorphisms as follows: The fundamental group of (X,p) is the most most general group G for which we have homomorphisms $\alpha : \pi_1(U,x_0) \to \pi_1(X,x_0)$. and $\beta : \pi_1(V,x_0) \to \pi_1(X,x_0)$ such that $\beta \circ i_{0*} = \alpha \circ j_{0*}$. More precisely, if $i : U \to X$ and $j : V \to X$ denote the inclusion mappings, then given any $(G, \alpha\beta)$ as above, *there is a unique homomorphism*

$$\Phi : \pi_1(X,x_0) \ \longrightarrow \ G$$

such that $\alpha = \Phi \circ i_*$ *and* $\beta = \Phi \circ j_*$.

In the language of category theory, one says that the triple $(\pi_1(X,x_0), i_*, j_*)$ is the **pushout** of the diagram associated to $(\pi_1(U \cap V,x_0), i_{0*}, j_{0*})$; further information on pushouts is given in `pushouts.pdf`.

The proof that the images of $\pi_1(U,p)$ and $\pi_1(V,p)$ generate $\pi_1(X,p)$ is relatively easy compared to the proof that the kernel of the map from $\pi_1(U,p) * \pi_1(V,p)$ to $\pi_1(X)$ is generated by the elements described above, so we shall give an alternate proof when X satisfies a condition which holds for some of the most important examples of topological spaces.

Definition. An arcwise connected, locally arcwise connected topological space X is said to be *locally simply connected* if every point has an open neighborhood base of simply connected sets. Clearly this implies that X is semilocally simply connected and hence has a universal covering space. Our argument actually goes through if X is semilocally simply connected, but we make the stronger assumption in order to simplify the discussion.

In the course of our proof we shall need the following property of regular coverings:

LEMMA. *Let X be a connected and locally simply connected space, let $p : (E,e_0) \to (B,b_0)$ be a regular covering space projection, and let G be the associated group of covering transformations (which is isomorphic to the opposite group of $\pi_1(X)$). Then $p = h \circ q$ where $q : E \to E/G$ is the quotient projection and h is a homeomorphism.*

Since a universal (simply connected) covering is regular, the result applies to universal coverings.

This follows from the construction of universal coverings in Munkres. We shall give a proof not based upon the construction at the end of this document.

In the setting described above, let G and H denote the fundamental groups of U and V respectively, and let $\widetilde{U}$ and $\widetilde{V}$ denote their universal coverings. As before, let N be the normal subgroup of $G * H$ which is normally generated by elements of the form

$$i_{0*}(y) \cdot \left(i_{0*}(y) \right)^{-1}$$

where $y \in \pi_1(U \cap V, x_0)$ and $i_0 : U \cap V \to V$, $j_0 : U \cap V \to U$ are the inclusion mappings. We are then interested in the group

$$\Gamma \quad = \quad (G * H)/N$$

and since $j \circ i_0 = i \circ j_0$ we know that there is a canonical homomorphism from Γ to $\pi_1(X, x_0)$. Since the images of G and H generate the fundamental group of X, we know that this canonical homomorphism is onto. We shall prove that the map is 1–1 by constructing a covering space E of X whose fundamental group is isomorphic touch that the image of $\pi_1(E, e)$ is equal to the image of Γ.

Digression. The construction of the desired covering space involves some general concepts that are also important in other mathematical contexts.

NOTATIONAL CONVENTIONS. **1.** If G is a group then G^* will denote the opposite group with binary operation $g_1 \otimes g_2 = g_2 \cdot g_1$, where the latter means the original binary operation on G.

2. If we are given a group L which acts on a space X and a homomorphism $j : L \to M$, then we define $M \times_L X$ to be the quotient of $M \times X$ modulo the equivalence relation defined by $(g, x) \sim (g \cdot j(h), h^{-1} \cdot x)$ for all (g, x) in $M \times X$ and $h \in L$. It follows immediately that if $p : E \to B$ is a covering space projection (where E is *not necessarily connected*) and L acts as a group of covering space transformations on E, then $M \times_L E$ is a **not necessarily connected** covering space over B with projection p_M sending $[g, x]$ to $p(x)$ for all (g, x). This is often called a *balanced product construction.*

This construction has two important properties:

TRANSITIVITY PROPERY. *In the setting above, if we are also given a homomorphism $k : M \to N$, then there is a canonical homeomorphism from $N \times_M (M \times_L X)$ to $N \times_L X$. If $E \to B$ is a regular covering space and L is the group of covering space transformations on E, then the canonical homeomorphism is in fact an equivalence of regular covering spaces.*■

RESTRICTION PROPERY. *Suppose that $E \to B$ is a covering space projection such that E is simply connected and L acts on E by covering space transformations. Suppose that A is an arcwise connected, locally arcwise connected subspace of B which has a simply connected covering space, and let $j_* : \pi_1(A, b_0) \to \pi_1(B, b_0)$ be induced by the inclusion of A in B. If E_A is the inverse image of A in E and $p_A : E_A \to A$ is the restricted covering space (which need not be connected), then there is an equivalence of covering spaces from E_A to $\pi_1(B, b_0)^* \times_{\pi_1(A, b_0)^*} \widetilde{A}$, where as usual $\widetilde{A} \to A$ denotes the universal covering space projection.*■

Both proofs are straightforward and left as exercises; in each case one needs to use the previously stated lemma on regular coverings.

With these concepts at our disposal, we may complete the proof of the "hard" part of the Seifert-van Kampen Theorem as follows:

Let $G \cong \pi_1(U)^*$ and $H \cong \pi_1(V)^*$ denote the deck transformation groups for the universal coverings $\widetilde{U}$ and $\widetilde{V}$ respectively, and consider the spaces

$$U_{\Gamma^*} \quad = \quad \Gamma^* \times_G \widetilde{U} \ , \qquad V_{\Gamma^*} \quad = \quad \Gamma^* \times_H \widetilde{V} \ .$$

By the Transitivity and Restriction Properties, the restrictions of these covering spaces to $U \cup V$ are canonically equivalent to

$$\Gamma^* \times_{\pi_1(U \cap V)^*} (U \cap V)^\sim$$

(where $(\ )^\sim$ denotes the universal covering space), and if we take the quotient of $U_{\Gamma^*} \amalg V_{\Gamma^*}$ formed by identifying points in these two open subsets via the equivalence of covering spaces, we obtain a space E, a covering space projection $E \to X$, and an action of Γ^* on E by covering space transformations. By construction, this action is transitive on the inverse image F of the base point p; in other words, if $e_0 \in F$ is the base point of E and $e_1 \in E$, then there is a (necessarily unique) covering transformation $T \in \Gamma^*$ such that $T(e_0) = e_1$.

CLAIM: The space E is arcwise connected.

In fact, it suffices to show that the inverse image of F lies in a single arc component of E, for if $y \in E$ then one has a continuous curve γ joining $p(y) \in X$ to the base point of X, and if we take the unique lifting of γ which starts at y we obtain a curve joining y to a point in F; if all of F lies in a single arc component of E, it then follows that every point of E lies in this arc component.

By construction, the points of F are in 1–1 correspondence with the elements of the pushout group Γ^*, and given two points in F there is a unique element of $T \in \Gamma^*$ sending the first to the second.

Let e_0 denote some chosen basepoint of E which maps to the basepoint of X. We shall first check that e_0 and $T(e_0)$ lie in the same component of E if T lies in the image of the fundamental group of U or the fundamental group of V; more correctly, we shall only consider the case where T comes from the first group, since the proof in the second case follows by systematically replacing U by V throughout the discussion. Let $k_U : \widetilde{U} \to E$ be the mapping given by the construction of E, and let u_0 denote the base point of $\widetilde{U}$, so that k_U maps u_0 to e_0. Suppose that $T(e_0) = e_0 \cdot g$, where $g \in \Gamma^*$ comes from $g' \in G$, and let α be a based closed curve in U representing g'. If $\widetilde{\alpha}$ is the unique lifting of α starting at u_0, then it follows that $T(e_0) = k_U \circ \widetilde{\alpha}(1)$, which means that $T(e_0)$ and e_0 lie in the same arc component of E. If S is an arbitrary covering transformation of E, then it also follows that $S(e_0)$ and $S \circ T(e_0)$ lie in the same arc component of E. As noted before, similar considerations hold when $T(e_0) = e_0 \cdot h$, where $h \in \Gamma^*$ comes from $h' \in H$.

Given the conclusions of the preceding paragraph, one can use the fact that the images of $G = \pi_1(U)^*$ and $H = \pi_1(V)^*$ generate Γ^* to deduce that every point in F lies in the same component as e_0 and hence E is arcwise connected. Specifically, if $T \in \Gamma^*$ then we may write T as a composite $T = T_1 \ \cdots \ T_k$ where each T_i is either in the image of G or the image of H; if P_i denotes the product of the first i factors with P_0 equal to the identity, then by the preceding paragraph we know that $P_i(e_0)$ and $P_{i-1}(e_0)$ lie in the same arc component of E. Combining these, we conclude that $T(e_0) = P_k(e_0)$ and $e_0 = P_0(e_0)$ lie in the same arc component of E; since every point in F has the form $T(e_0)$ for some T, it follows that all of F lies in the same arc component of E, and as noted before this implies that E itself must be arcwise connected.

Since Γ^* acts as a group of covering transformations on E and it is transitive on F, the results about the action of $\pi_1(E)$ on F imply that the image J of $\pi_1(E, e_0)$ in $\pi_1(B, b_0)$ is a normal subgroup and the quotient group is isomorphic to Γ. In fact, the projection map $\partial : \pi_1(B, b_0) \to \Gamma$ is given by taking a closed curve γ representing an element g of the fundamental group of B, forming the unique lifting $\widetilde{\gamma}$ starting at e_0, and defining $\partial(g)$ so that $\widetilde{\gamma}(1) = g \cdot e_0$. One must use the fact that J is normal in the fundamental group to prove that ∂ is a homomorphism.

Combining this with previous observations, we obtain the diagram of morphisms displayed below, in which the square is commutative (all compositions of morphisms between two objects in this part of the diagram are equal).

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d, "J(U)"] & & \\
\pi_1(V) \arrow[r, "J(V)"] & \Gamma \arrow[r, "\Phi"] & \pi_1(X) \arrow[r, "\partial"] & \Gamma
\end{tikzcd}$$

The map Φ is the homomorphism given by the universal mapping property of the pushout group Γ. If we can show that $\partial \circ \Phi$ is the identity, then it will follow that Φ is injective. Since we already know that Φ is surjective, it will follow that Φ is an isomorphism, and the proof will be complete.

The key point is to prove that the composites

$$\pi_1(U) \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow \Gamma \qquad\qquad \pi_1(V) \longrightarrow \Gamma \longrightarrow \pi_1(X) \longrightarrow \Gamma$$

are just the standard maps $J(U)$ and $J(V)$ from $\pi_1(U)$ and $\pi_1(V)$ into the pushout Γ. Since the identity 1_Γ on Γ satisfies $1_\Gamma \circ J(U) = J(U)$ and $1_\Gamma \circ J(V) = J(V)$, It follows that the identity and $\partial \circ \Phi$ agree on the images of $J(U)$ and $J(V)$. Since these sets generate Γ, it follows that $\partial \circ \Phi = \text{identity}_\Gamma$, and as noted above this suffices to complete the proof of the Seifert-van Kampen Theorem.

It will be helpful to let i_{U*} and i_{V*} denote the maps of fundamental groups induced by the inclusions of U and V in X; by construction we have $i_{U*} = \Phi \circ J(U)$ and $i_{V*} = \Phi \circ J(V)$.

As before, it suffices to show that $\partial \circ \Phi \circ J(U) = J(U)$, for the argument in the other case will follow by systematic substitution of V for U throughout. — Let h' be an element in $\pi_1(U)$, and let γ be its image in Γ. By construction, the covering space transformation determined by $\partial \circ \Phi(\gamma) \in \Gamma$ sends the base point e_0 to $e_0 \cdot i_{U*}(h') = e_0 \cdot \Phi(\gamma)$. On the other hand, we also know that the covering space transformation of $\widetilde{U}$ associated to h' sends u_0 to $u_0 \cdot h'$, and if we apply the mapping k_U from the previous discussion, it follows that the covering space transformation of E associated to $\Phi(\gamma) = i_{U*}(h')$ sends $e_0 = k_U(u_0)$ to $e_0 \cdot i_{U*}(h') = e_0 \cdot \Phi(\gamma)$.

The preceding argument shows that $\partial \circ i_{U*} = J(U)$, and the identity in the first sentence of the preceding paragraph then follows because $i_{U*} = J(U) \circ \Phi$. As noted above, we have a similar identity involving V. Taken together, these imply that the restrictions of $\partial \circ \Phi$ to the images of $J(U)$ and $J(V)$ are the identity, and since these images generate Γ it follows that $\partial \circ \Phi$ must be the identity, as claimed.■

Remark. In fact, the covering space E constructed in the proof is simply connected. This will follow if we can show that ∂ is an isomorphism, for the latter will imply that the kernel of ∂ — which is isomorphic to the fundamental group of E — must be trivial; to see the assertion regarding ∂, note that the proof implies that Φ is an isomorphism, and since $\partial \circ \Phi$ is the identity it follows that $\partial = \Phi^{-1}$.■

Appendix: A lemma on regular coverings

We shall now give a proof of the lemma stated at the beginning of this document.

Previous results imply that the group G of covering transformations for the regular covering $p : E \to X$ is isomorphic to $\pi_1(X)/(\text{Image } p_*)$ with reversed multiplication, and if $e \in E$ is an arbitrary point such that $p(e) = b_0$ then there is a unique deck transformation T such that $T(e_0) = e$. Since a deck transformation T satisfies $p \circ T = p$, it follows that if $y' = T(y)$ in E then $p(y) = p(y')$, which in turn implies that the covering space projection $p : E \to B$ has a factorization $p = h \circ q$, where q is the quotient projection from E to E/G. We need to show that h is a homeomorphism.

By construction the mapping h is continuous and onto (since p is onto), so we only need to prove that (i) the mapping h is open, (ii) the mapping h is 1–1.

To see that h is open, let V be open in E/G so that $q^{-1}[V]$ is open in E. We then have

$$h[V] \quad = \quad p\left[\,q^{-1}[V]\,\right]$$

and since p is open it follows that h is also open.

To prove that h is 1–1, suppose that $z_1, z_2 \in E/G$ are such that $h(z_1) = h(z_2)$; if we choose $e_i \in E$ such that $q(e_i) = z_i$, then the injectivity of h reduces to checking that $q(e_1) = q(e_2)$ or equivalently that there is a deck transformation T such that $T(e_1) = T(e_2)$. — To see this, first join e_0 to e_1 by some curve α. The assumptions imply that $p(e_1) = p(e_2)$, and therefore there is a unique lifting β_0 of $-p \circ \alpha$ with initial point e_2. By construction we know that $\beta_0(1)$ lies in the inverse image of $\{b_0\}$. Since we have a regular covering, there is a unique deck transformation T such that $T(e_0) = \beta_0(1)$.

Now let $\beta = -\beta_0$, so that β and $T \circ \alpha$ are both liftings of $p \circ \alpha$ starting at e_0 and hence their endpoints are equal. Since $\beta(1) = e_2$ and $T \circ \alpha(1) = T(e_1)$, it follows that $T(e_1) = e_2$, which is what we need to prove in order to show that h is a homeomorphism.■