Mathematics 205B, Winter 2014, Take–Home Assignment

This will be due on **Friday, March 7, 2014, at 12:10 P.M.** at the beginning class or by prior arrangement in my mailbox or at the front desk of Surge 202 at the same time. If you wish to use some version of TEX in writing up your answers, please feel free to do so. *You must show the work behind or reasons for your answers.*

1. Suppose that $\mathbf{K}$ is a connected simplicial complex which is the union of two connected subcomplexes $\mathbf{K}_1$ and $\mathbf{K}_2$ such that $\mathbf{K}_1 \cap \mathbf{K}_2$ is star-shaped with respect to the vertex $\mathbf{v}$ (which we shall assume comes first in the ordering of vertices).

PROVE that

$$H_q(\mathbf{K}) \quad \cong \quad H_q(\mathbf{K}_1) \ \oplus \ H_q(\mathbf{K}_2)$$

for all $q > 0$, and describe the difference between the left and right hand sides if $q = 0$.

2. Suppose that $\mathbf{K}$ is a connected simplicial complex which is the union of two connected subcomplexes $\mathbf{K}_1$ and $\mathbf{K}_2$.

PROVE that $H_1(\mathbf{K})$ is generated by the images of $H_1(\mathbf{K}_1)$ and $H_1(\mathbf{K}_2)$ (with respect to the maps induced by inclusion of chain complexes) if and only if $\mathbf{K}_1 \cap \mathbf{K}_2$ is connected.

Note. Half of this result is a partial analog of the van Kampen Theorem for fundamental groups, and the other half is an anlog of a companion result: If in the statement of the van Kampen Theorem one removes the hypothesis that the intersection be arcwise connected, then the conclusion is systematically false; in fact, whenever the intersection is not arcwise connected, then the images of $\pi_1(U)$ and $\pi_1(V)$ generate a proper subgroup of $\pi_1(X)$ with infinite index in the latter.

3. Suppose that we are given a simplex in $\mathbb{R}^2$ with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$. Let

$$\mathbf{z} \quad = \quad \frac{1}{3}\left(\mathbf{a} + \mathbf{b} + \mathbf{c}\right)$$

be the barycenter of the simplex, and let $\mathbf{x}_\pm \in \mathbb{R}^3$ denote the point $(\mathbf{z}, \ \pm 1)$. Consider the simplicial complex L_0 which is the union of the six 2-simplices $\mathbf{x}_\pm \, \mathbf{a}\, \mathbf{b}$, $\mathbf{x}_\pm \, \mathbf{a}\, \mathbf{c}$ and $\mathbf{x}_\pm \, \mathbf{b}\, \mathbf{c}$, and suppose that we form the following larger simplicial complexes from $\mathbf{L}_0$:

$\mathbf{L}_1$ is formed by adding the 2-simplex $\mathbf{abc}$.

$\mathbf{L}_2$ is formed by adding the 1-simplex $\mathbf{x}_-\mathbf{x}_+$.

COMPUTE the homology groups of $\mathbf{L}_i$, where $i = 0, 1, 2$. [*Hints:* The subcomplexes $\mathbf{H}_-$ and $\mathbf{H}_+$ consisting of all 2-simplices in $\mathbf{L}_0$ which have $\mathbf{x}_-$ or $\mathbf{x}_+$ (respectively) as a vertex are star-shaped. What is their intersection?]