Solutions to written assignment due February 1, 2012

1. $\boxed{A}$ Assume Y is locally simply connected.

$H = \text{Image } p_* \subseteq \pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y) \cong \pi_1(Y)$.

If Y is locally simply connected, then there is a connected covering space $q: W' \to Y$ such that $\text{Image } q_* \leftrightarrow \text{Image } p_*$ under the iso $\pi_1(X \times Y) \cong \pi_1(Y)$.

But now $1_X \times q : X \times W' \to X \times Y$ will be a connected covering space such that $\text{Image } (1_X \times q)_* = \text{Image } p_*$ so by the classification result these covering spaces are equivalent; i.e. there is a homeo $h: W \to X \times W'$ such that $p = (1_X \times q) \circ h$. ∎

$\boxed{B}$ The result remains valid without the local simple connectivity assumption. It is enough to find a connected covering space $W' \xrightarrow{q} Y$ such that Image $q_* \leftrightarrow$ Image p_*.

Let (x_0, y_0) be the base point in $X \times Y$ and let $W' = p^{-1}[\{x_0\} \times Y]$. Then $W' \to \{x_0\} \times Y \cong Y$ is a covering space.* We claim that the image of $\pi_1(W') \to \pi_1(Y)$ corresponds to the image of p_* AND (important!) W' is connected. Note that $\pi_1(Y) \xrightarrow{\cong} \pi_1(X \times Y)$.

(1) W' is connected. Let F = fiber (x_0, y_0), so that $F \subseteq W' \subseteq W$. As in the proof of the Seifert-van Kampen Thm., it suffices to show that every point in F can be joined to a base-point inside of W' (we know there is a joining curve in W, and we want one in W').

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* In general p^{-1}[subset] is a covering, but we do not know if it is a connected covering, and in fact it usually is not connected (see the HW exercises).

But if γ is a curve in W joining a point in F to the base point, then $p\circ\gamma$ is a closed curve in $X\times Y$ $\Rightarrow$ it can be deformed base point preservingly to a ~~point~~ curve in $\{x_0\}\times Y$. By the Covering Homotopy property, this means γ can be deformed in an endpoint preserving fashion to a curve in W'; and this is what we need.

(2) The image of $\pi_1(W')$ in $\pi_1(Y)$ equals Image p_* under the correspondence $\pi_1(Y) \leftrightarrow \pi_1(X\times Y)$. Since these groups are isomorphic via the homomorphism associated to $\{x_0\}\times Y \subseteq X\times Y$, it follows that the action of $\pi_1(Y)$ on $F\subseteq W'$ corresponds to the action of $\pi_1(X\times Y)$ on $F\subseteq W$. In particular, the image subgroups, which are the stabilizers of the base point, must also correspond bijectively. Therefore the image of $\pi_1(W')$ in $\pi_1(Y)$ [the W' stabilizer] corresponds to Image p_* [the W stabilizer] which is what we needed to prove. ∎

$\mathbb{R}_+$ = positive reals

2. Let $h : \mathbb{C}-\{0\} \longrightarrow S^1 \times \mathbb{R}_+$ be the homeomorphism $z \longrightarrow \left(\frac{z}{|z|}, |z|\right)$, and let $\Psi : \mathbb{C}-\{0\} \longrightarrow \mathbb{C}-\{0\}$ be the n-th power map. If $H : \mathbb{C}-\{0\} \longrightarrow S^1 \times \mathbb{R}_+$ is the homeomorphism $\left(\frac{z}{|z|}, \sqrt[n]{|z|}\right) \longleftrightarrow z$ and $\Phi : S^1 \times \mathbb{R}_+ \longrightarrow S^1 \times \mathbb{R}_+$ is the covering space projection $(z, t) \longrightarrow (z^n, t)$, then there is a commutative diagram

$$\begin{tikzcd} \mathbb{C}-\{0\} \arrow[r, "H", "\cong"'] \arrow[d, "\Psi"'] & S^1 \times \mathbb{R}_+ \arrow[d, "\Phi"] \\ \mathbb{C}-\{0\} \arrow[r, "h", "\cong"'] & S^1 \times \mathbb{R}_+ \end{tikzcd}$$

and this implies that Ψ is a regular n-sheeted covering space projection (since Φ is).

Now suppose $U \subseteq \mathbb{C}-\{0\}$ is simply connected. Then we want a lifting W such that

$$\begin{tikzcd} & \mathbb{C}-\{0\} \arrow[d, "\Psi"] \\ U \arrow[ur, dashed, "W"] \arrow[r, "j"', "\text{INCLUSION}"'] & \mathbb{C}-\{0\} \end{tikzcd}$$

commutes.

By the Lifting Criterion, there is such a lifting because U simply connected $\Rightarrow$ j_* is trivial $\Rightarrow$ Image $j_* \subseteq$ Image Ψ_*. ∎