Mathematics 205B, Winter 2012, Take-home assignment 2

This will be due on **Monday, March 19, 2012, at 9:00 A.M.** at the beginning of the final examination period. If you wish to use some version of TEX in writing up your answers, please feel free to do so. *Unless explicitly stated otherwise, you must show the work behind or reasons for your answers.*

1. (*a*) Suppose that the simplicial complex $\mathbf{K}$ is a union of two connected subcomplexes $\mathbf{K}_1 \cup \mathbf{K}_2$ where each $\mathbf{K}_i$ is connected, and suppose also that the intersection $\mathbf{K}_1 \cap \mathbf{K}_2$ is starshaped with respect to some vertex $\mathbf{v}$, where $\mathbf{v}$ is minimum in a given linear ordering ω of the vertices in $\mathbf{K}$. Prove that

$$H_q(\mathbf{K}) \quad \cong \quad H_q(\mathbf{K}_1) \;\oplus\; H_q(\mathbf{K}_2)$$

for all $q > 0$.

(*b*) Given a simplicial complex $\mathbf{K}$, prove that it is isomorphic to a subcomplex of a complex $\mathbf{L}$ such that the homology homomorphisms $H_q(\mathbf{K}) \to H_q(\mathbf{L})$ is trivial for all $q > 0$. [*Hint:* Why is $\mathbf{K}$ isomorphic to a subcomplex of some simplex with the same number of vertices?]

2. Let n be a positive integer. A *topological n-manifold* is a Hausdorff space M such that every point $p \in M$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

(*a*) Prove that the local homology groups of M at each point $x \in M$ are infinite cyclic in dimension n and zero otherwise. [*Hint:* Use the localization principle for local homology and the fact that x has an open neighborhood homeomorphic to a subset of $\mathbb{R}^n$.]

(*b*) Prove that if M is a topological m-manifold and N is a topological n-manifold. Then M is homeomorphic to N only if $m = n$.

(*c*) Suppose that $f : M \to N$ is a 1–1 continuous mapping of topological n-manifolds. Prove that f is an open mapping. [*Hint:* Why does it suffice to prove that each $p \in M$ has an open neighborhood U_p such that $f|U_p$ is 1–1? Each point $f(p)$ has an open neighborhood V_p which is homeomorphic to an open subset of $\mathbb{R}^n$. Why is there a neighborhood of p which is also homeomorphic to an open subset of $\mathbb{R}^n$ and is mapped into V_p by f?]

(*d*) Prove that S^n is not homeomorphic to a subset of $\mathbb{R}^n$. — In nonmathematical terms, this means that one cannot continuously flatten out a deflated beach ball on a table without some overlapping of points.

3. Let (A_*, d_*) be a chain complex (say over the category of abelian groups). A **multiplicative structure** on (A_*, d_*) is a family of bilinear mappings

$$\varphi_{p,q} : A_p \times A_q \to A_{p+q}$$

which is a homomorphism in each variable with the other held constant and satisfies the following version of the Leibniz rule:

$$d\varphi(a_p, a_q) \quad = \quad \varphi(d(a_p), a_q) \;+\; (-1)^p \varphi(a_p, d(a_q))$$

Usually it is convenient to denote $\varphi(x, y)$ by notation such as $x * y$.

(a) Prove that φ induces a family of bilinear mappings

$$\varphi_* : H_p(A) \times H_q(A) \to H_{p+q}(A)$$

such that if u and v are represented by cycles x and y, then $x*y$ is a cycle and $u*v$ is represented by $x * y$. The proof should include justifications of the following assertions (this list is not necessarily exhaustive):

(1) If x and y are cycles then so is $x * y$.

(2) If $x = dw$ and y is a cycle then $x * y$ is a boundary. Likewise, if x is a cycle and $y = dv$ then $x * y$ is a boundary.

(c) Prove that the multiplicative structure in homology satisfies the associative law $(u*v)*w = u * (v * w)$ if the multiplicative structure on the chain complex level has this property.

(d) A two-sided unit for a multiplicative structure is a class $e \in A_0$ such that $de = 0$ and $e * a = a = a * e$ for all a. Prove that the homology class of e is a two-sided unit for the multiplicative structure in homology and that this class is nontrivial if $H_q(A) \neq 0$ for some $q \neq 0$.