from 9.7. Theorems 9.3 and 4 are proved in a similar way using the reduced H.S. of (X,A) instead of the H.S. of (X,A). The exactness of the reduced H.S. was proved in 8.6.

DEFINITION 9.1c. A space X is said to be *cohomologically trivial* if $H^q(X) = 0$ for $q \neq 0$ and $\tilde{H}^0(X) = 0$. If A is nonvacuous, (X,A) is said to be *cohomologically trivial* if $H^q(X,A) = 0$ for all q.

THEOREM 9.2c. *If (X,A) is cohomologically trivial and A is nonvacuous, then i^*: $H^q(X) \approx H^q(A)$ for all q. Conversely, if these relations hold, (X,A) is cohomologically trivial.*

THEOREM 9.3c. *If A is cohomologically trivial and is a nonvacuous subset of X, then*

$$\begin{aligned} j^*:&\ H^q(X,A) \approx H^q(X) \qquad &\textit{for } q \neq 0,\\ \bar{j}^*:&\ H^0(X,A) \approx \tilde{H}^0(X). \end{aligned}$$

Conversely, if these relations hold, A is cohomologically trivial.

THEOREM 9.4c. *If X is cohomologically trivial and A is a nonvacuous subset, then*

$$\begin{aligned} \delta:&\ H^{q-1}(A) \approx H^q(X,A) \qquad &\textit{for } q \neq 1,\\ \bar{\delta}:&\ \tilde{H}^0(A) \approx H^1(X,A). \end{aligned}$$

Conversely, if these relations hold, X is cohomologically trivial.

COROLLARY 9.5c. *If both X and A are cohomologically trivial, so also is (X,A).*

10. THE HOMOLOGY SEQUENCE OF A TRIPLE (X,A,B)

Suppose that $X \supset A \supset B$ and that the inclusions

$$\bar{i}:\ (A,B) \subset (X,B), \qquad \bar{j}:\ (X,B) \subset (X,A)$$

are admissible. Then (X,A,B) is called an admissible *triple*. The inclusion maps and boundary operators associated with the pairs $(X,A),(X,B),(A,B)$ are denoted by

$$\begin{aligned} i&: A \to X, & j&: X \to (X,A), & \partial&: H_q(X,A) \to H_{q-1}(A),\\ i'&: B \to X, & j'&: X \to (X,B), & \partial'&: H_q(X,B) \to H_{q-1}(B),\\ i''&: B \to A, & j''&: A \to (A,B), & \partial''&: H_q(A,B) \to H_{q-1}(B). \end{aligned}$$

A comparison of these maps with $\bar{i}$ and $\bar{j}$ suggests

DEFINITION 10.1. $\bar{\partial} = j''_*\partial$ is called the *boundary* operator of the triple (X,A,B):

$$\bar{\partial}:\ H_q(X,A) \to H_{q-1}(A,B).$$

The lower sequence of groups

$$\cdots \leftarrow H_{q-1}(A,B) \xleftarrow{\bar{\partial}} H_q(X,A) \xleftarrow{\bar{j}_*} H_q(X,B) \xleftarrow{\bar{i}_*} H_q(A,B) \leftarrow \cdots$$

is called the *homology sequence* of the triple (X,A,B). It is indexed so that $H_0(X,A)$ is the 0^{th} group.

THEOREM 10.2. *The homology sequence of a triple is exact.*

The proof of this theorem is a lengthy and tricky exercise in the use of axioms 1, 2, 3, and 4. The reader may save himself the effort of following the proof if he is willing to replace the Exactness axiom by the stronger proposition 10.2. It reduces to the former when $B = 0$.

PROOF. Reference to the diagram below will assist the reader in following the proof.

$$\begin{array}{ccccccc}
H_q(A) & \xrightarrow{i_*} & H_q(X) & & & & \\
\downarrow{\scriptstyle j''_*} & & \downarrow{\scriptstyle j'_*} & & & & \\
H_q(A,B) & \xrightarrow{\bar{i}_*} & H_q(X,B) & \xrightarrow{\bar{j}_*} & H_q(X,A) & & \\
 & & \downarrow{\scriptstyle \partial'} & & \downarrow{\scriptstyle \partial} & & \\
 & & H_{q-1}(B) & \xrightarrow{i''_*} & H_{q-1}(A) & \xrightarrow{i_*} & H_{q-1}(X) \\
 & & & & \downarrow{\scriptstyle j''_*} & & \downarrow{\scriptstyle j'_*} \\
 & & & & H_{q-1}(A,B) & \xrightarrow{\bar{i}_*} & H_{q-1}(X,B)
\end{array}$$

Commutativity relations hold in each square of the diagram. In addition to the homomorphisms displayed in the diagram, we have:

$$j_* = \bar{j}_* j'_*, \qquad i'_* = i_* i''_*, \qquad \partial'' = \partial' \bar{i}_*, \qquad \bar{\partial} = j''_* \partial.$$

The proof breaks up into the proofs of six propositions:

(1) $\bar{j}_* \bar{i}_* = 0$, (2) $\bar{\partial} \bar{j}_* = 0$, (3) $\bar{i}_* \bar{\partial} = 0$.

(4) If $x \in H_q(X,B)$ and $\bar{j}_* x = 0$, there exists an $x' \in H_q(A,B)$ such that $\bar{i}_* x' = x$.

(5) If $y \in H_q(X,A)$ and $\bar{\partial} y = 0$, there exists a $y' \in H_q(X,B)$ such that $\bar{j}_* y' = y$.

(6) If $z = H_{q-1}(A,B)$ and $\bar{i}_* z = 0$, there exists a $z' \in H_q(X,A)$ such that $\bar{\partial} z' = z$.

PROOF OF (1). The map $\bar{j}\bar{i}$: $(A,B) \to (X,A)$ can be expressed as the composition of the inclusion maps k: $(A,B) \subset (A,A)$ and l:

$(A,A) \subset (X,A)$. By 8.1, $H_q(A,A) = 0$. Therefore $k_* = 0$. It follows that $\bar{j}_*\bar{i}_* = (\bar{j}\bar{i})_* = (lk)_* = l_*k_* = 0$.

PROOF OF (2). $\bar{\partial}\bar{j}_* = j''_*\partial\bar{j}_* = j''_*i''_*\partial'$. Since the H.S. of (A,B) is exact, $j''_*i''_* = 0$.

PROOF OF (3). $\bar{i}_*\bar{\partial} = \bar{i}_*j''_*\partial = j'_*i_*\partial$. Since the H.S. of (X,A) is exact, $i_*\partial = 0$.

PROOF OF (4). Since $\bar{j}_*x = 0$, it follows that

$$i''_*\partial'x = \partial\bar{j}_*x = 0.$$

By the exactness of the H.S. of (A,B), the element $\partial'x$ of the kernel of i''_* is the image of an element

$$x_1 \varepsilon H_q(A,B), \qquad \partial''x_1 = \partial'x.$$

Then

$$\partial'(x - \bar{i}_*x_1) = \partial'x - \partial'\bar{i}_*x_1 = \partial'x - \partial''x_1 = 0.$$

By the exactness of the H.S. of (X,B), the element $x - \bar{i}_*x_1$ of the the kernel of ∂' is the image of an element

$$u \varepsilon H_q(X), \qquad j'_*u = x - \bar{i}_*x_1.$$

Then

$$j_*u = \bar{j}_*j'_*u = \bar{j}_*(x - \bar{i}_*x_1) = \bar{j}_*x - \bar{j}_*\bar{i}_*x_1 = 0$$

by (1) and hypothesis. By exactness of the H.S. of (X,A), the element u of the kernel of j_* is the image of an element

$$v \varepsilon H_q(A), \qquad i_*v = u.$$

Define

$$x' = x_1 + j''_*v.$$

Then

$$\begin{aligned} \bar{i}_*x' &= \bar{i}_*x_1 + \bar{i}_*j''_*v = \bar{i}_*x_1 + j'_*i_*v \\ &= \bar{i}_*x_1 + j'_*u = \bar{i}_*x_1 + x - \bar{i}_*x_1 = x. \end{aligned}$$

PROOF OF (5). Since

$$j''_*\partial y = \bar{\partial}y = 0,$$

by the exactness of the H.S. of (A,B), the element ∂y of the kernel of j''_* is the image of an element

$$u \varepsilon H_{q-1}(B), \qquad i''_*u = \partial y.$$

Then, by exactness of the H.S. of (X,A), $i_*\partial = 0$. So

$$i'_*u = i_*i''_*u = i_*\partial y = 0.$$

By exactness of the H.S. of (X,B), the element u of the kernel of i'_* is the image of an element

$$y_1 \in H_q(X,B), \qquad \partial' y_1 = u.$$

Then

$$\partial(y - \bar{j}_* y_1) = \partial y - \partial \bar{j}_* y_1 = \partial y - i''_* \partial' y_1 = \partial y - i''_* u = 0.$$

By exactness of the H.S. of (X,A), the element $y - \bar{j}_* y_1$ of the kernel of ∂ is the image of an element

$$v \in H_q(X), \qquad j_* v = y - \bar{j}_* y_1.$$

Define

$$y' = y_1 + j'_* v.$$

Then

$$\bar{j}_* y' = \bar{j}_* y_1 + \bar{j}_* j'_* v = \bar{j}_* y_1 + j_* v = \bar{j}_* y_1 + y - \bar{j}_* y_1 = y.$$

Proof of (6). Since $\bar{i}_* z = 0$, it follows that

$$\partial'' z = \partial' \bar{i}_* z = 0.$$

By exactness of the H.S. of (A,B), the element z of the kernel of ∂'' is the image of an element

$$u \in H_{q-1}(A), \qquad j''_* u = z.$$

Then

$$j'_* i_* u = \bar{i}_* j''_* u = \bar{i}_* z = 0.$$

By exactness of the H.S. of (X,B), the element $i_* u$ of the kernel of j'_* is the image of an element

$$v \in H_{q-1}(B), \qquad i'_* v = i_* u.$$

Since $i_* i''_* = i'_*$, it follows that

$$i_*(u - i''_* v) = i_* u - i_* i''_* v = i_* u - i'_* v = 0.$$

By exactness of the H.S. of (X,A), the element $u - i''_* v$ of the kernel of i_* is the image of an element

$$z' \in H_q(X,A), \qquad \partial z' = u - i''_* v.$$

Then

$$\bar{\partial} z' = j''_* \partial z' = j''_*(u - i''_* v) = j''_* u - j''_* i''_* v = j''_* u = z.$$

This completes the proof.

A *map* $f\colon (X_1,A_1,B_1) \to (X,A,B)$ of one triple into another is a map of X_1 into X which carries A_1 into A, and B_1 into B. The map f defines maps

$$f_1\colon (X_1,A_1) \to (X,A), \qquad f_2\colon (A_1,B_1) \to (A,B),$$
$$f_3\colon (X_1,B_1) \to (X,B).$$

The map f is *admissible* if f_1, f_2, and f_3 are admissible. In this case, the induced homomorphisms f_{1*}, f_{2*}, and f_{3*} map the groups of the H.S. of (X_1,A_1,B_1) into those of the H.S. of (X,A,B).

THEOREM 10.3. *A map $f\colon (X_1,A_1,B_1) \to (X,A,B)$ induces a homomorphism of the homology sequence of (X_1,A_1,B_1) into the homology sequence of (X,A,B).*

The proof requires only the verification of three commutativity relations and is left as an exercise for the reader.

THEOREM 10.4. *If B is nonvacuous, and any one of the three pairs $(X,A),(X,B),(A,B)$ is homologically trivial, then the homology groups of the remaining two pairs are isomorphic under the maps of the H.S. of (X,A,B). Explicitly:*

(1) *If $H_q(X,A) = 0$ for each q, then $\bar{i}_*\colon H_q(A,B) \approx H_q(X,B)$ for each q.*
(2) *If $H_q(X,B) = 0$ for each q, then $\bar{\partial}\colon H_q(X,A) \approx H_{q-1}(A,B)$ for each q.*
(3) *If $H_q(A,B) = 0$ for each q, then $\bar{j}_*\colon H_q(X,B) \approx H_q(X,A)$ for each q.*

Conversely, any one of the three conclusions implies the corresponding hypothesis.

The proof is the same as that of 9.2 except for notation.

THEOREM 10.5. *Let (X,A,B) be an admissible triple. If $B \subset A$ induces isomorphisms $H_q(B) \approx H_q(A)$ for all values of q, then $(X,B) \subset (X,A)$ also induces isomorphisms $H_q(X,B) \approx H_q(X,A)$ for all q. Similarly, if $A \subset X$ induces $H_q(A) \approx H_q(X)$ for all q, then $(A,B) \subset (X,B)$ induces $H_q(A,B) \approx H_q(X,B)$ for all q.*

PROOF. For the first assertion, the hypothesis and the exactness of the H.S. of (A,B) imply that $H_q(A,B) = 0$ for all q. The result follows now from 10.4.

Another proof is obtained by applying 4.2 to the inclusion map $(X,B) \subset (X,A)$.

The proof of the second part of the theorem is similar to that of the first part.