SOLUTIONS TO aabUpdate07.145A.w19.pdf

1. CORRECTION. The first sentence's assumption should read $H_q(X,A) = 0$ for $q \leq n$. $\boxed{\text{not } q < n}$

If $q < n$ then there is an exact sequence

$$0 = H_{q+1}(X,A) \to H_q(A) \xrightarrow{i_*} H_q(X) \to H_q(X,A) = 0$$

Therefore Kernel $i_* = 0 = H_q(X)/\text{Image } i_*$ which means i_* is an isomorphism.

If $q = n$ we only have an exact sequence

$$H_q(A) \xrightarrow{i_*} H_q(X) \to H_q(X,A) = 0$$

which means that $H_q(A) \to H_q(X)$ is onto. ∎

2. ~~If $f: A \to A'$ and $H_q(B) \to H_q(B')$ are homomorphisms~~

The Mayer-Vietoris sequence has the form

$$\cdots H_{q+1}(C) \xrightarrow{\Delta} H_q(A \cap B) \xrightarrow{(i_{A*}, -i_{B*})} \begin{matrix} H_q(A) \\ \oplus \\ H_q(B) \end{matrix} \xrightarrow{j_*} H_q(C) \xrightarrow{\Delta} \cdots$$

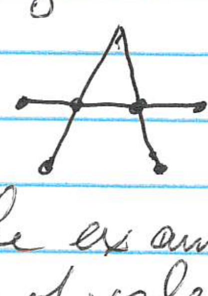
$$\begin{matrix} H_q(A) \oplus H_q(B) \\ \cong \\ H_q(A \oplus B) \end{matrix} \quad \text{(WHY?)}$$

This is the long exact homology sequence which comes from the following short exact sequence of chain complexes:

$$0 \longrightarrow (A\cap B)_* \xrightarrow[(i_A,\,-i_B)]{} \begin{matrix}A_*\\ \oplus\\ B_*\end{matrix} \xrightarrow{j} C_* \longrightarrow 0$$

Here is a proof that the latter exists: By assumption j is onto. Its kernel is all (a, b) so that $j(a) + j(b) = 0$, or $j(a) = -j(b)$. Since A_* and B_* are subcomplexes of C_*, this can only happen if $a = -b$ in C ~~for~~ or equivalently if $i_{A*}(a) = -i_{B*}(b)$ or there is $x \in (A\cap B)_*$ so that $i_{A*}(x) = -i_{B*}(x)$. ■

3. There are many examples. Here is one:



In each of the examples, there is at most one vertex of valence 4, but for the example above there are two such vertices. ■

Here is a more subtle example: The Cyrillic character

$$Ю$$

Like A, this has two valence 3 vertices. If these are removed from A, each component of the complement is homeo to $[0,1]$. If the valence 3 vertices are removed from Ю, one component's closure is homeo to S^1 while the other three closures are homeo to $[0,1]$. ∎

note there are 4 compts.

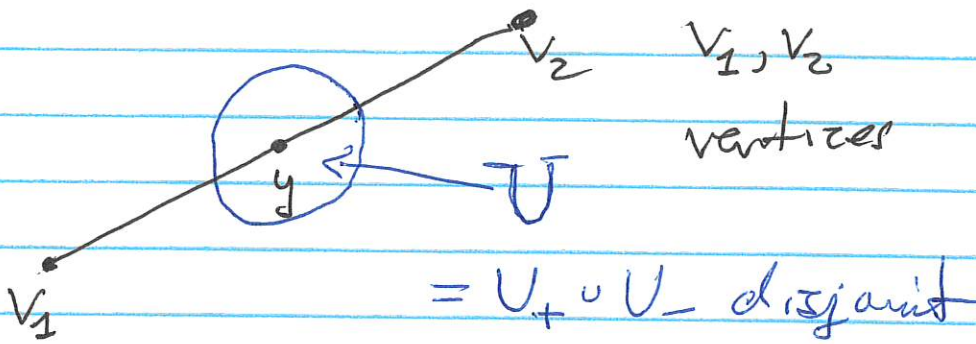
4. As in #2 this comes from a long exact homology sequence obtained from a short exact sequence of chain complexes. Let j denote the (onto) composite

$$(C/A)_* \xrightarrow[\text{proj onto quotient}]{} (C/A)_*/(B/A)_* \underset{\uparrow \text{ "Second Isomorphism Theorem"}}{\cong} (C/B)_*$$

The kernel of j is just the image of the inclusion $i: (B/A)_* \longrightarrow (C/A)_*$

so we have a short exact sequence
$0 \to (B/A)_* \to (C/A)_* \to (C/B)_* \to 0$ as desired. ∎

5. Here is a sketch



By the Jordan Curve Theorem, $\mathbb{R}^2 - P$ has two components whose frontiers are both P.

NOTE. Strictly speaking we only have shown the analogous result for $S^2 - P$, so here is a proof for $\mathbb{R}^2$. We know that $\infty \in S^2$ is ~~not~~ an interior point of $S^2 - P$, so the statement about frontiers will follow if we know that

$$i_{0*}\ \tilde{H}_0(\mathbb{R}^2 - P) \longrightarrow \tilde{H}_0(S^2 - P)$$

is an isomorphism. To see this, look at the long exact homology sequence for the pair $(S^2 - P, \mathbb{R}^2 - P)$. If $H_1 = H_0 = 0$ for this pair, then i_{0*} is an iso. But by excision and $S^2 = (S^2 - P) \cup \mathbb{R}$ we have an isomorphism

$$H_*(S^2 - P, \mathbb{R}^2 - P) \cong H_*(S^2, \mathbb{R}^2) = \begin{cases} \mathbb{Z} & * = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Back to #5 Write $\mathbb{R}^2 - P = V \cup W$ where V and W are disjoint open & connected, so that $P = \mathrm{Bdy}(V) = \mathrm{Bdy}(W)$. Therefore U contains points in both V and W.

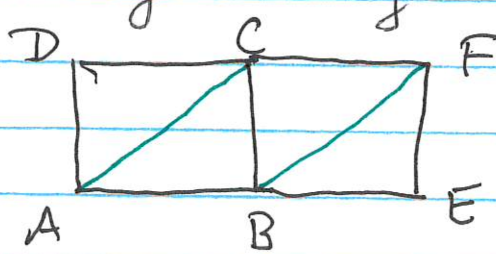
Also $U - P = (U \cap V) \cup (U \cap W) = U_+ \cup U_-$ implies that each of U_+, U_- is ~~contained~~ exactly one of ~~V, W~~ $(U \cap V)$, $(U \cap W)$. Hence

$U_+ \subseteq V$ and $U_- \subseteq W$ or vice versa. ∎

6. $S^2 \times \mathbb{R} \cong \mathbb{R}^3 - \{0\}$, open in $\mathbb{R}^3 \Rightarrow$
$S^2 \times \mathbb{R}^2 \cong (\mathbb{R}^3 - \{0\}) \times \mathbb{R} \cong$ open subset in $\mathbb{R}^4$.

Use invariance of dimension to conclude the spaces are not homeomorphic. ∎

7. Start by drawing a picture:



Add diagonals AC, BF to get a simplicial decomposition

The goal is to find a 2-chain such that the boundary terms involving AC and BF cancel.

Note that

$$d(ABC) = AB + BC - AC$$

$$d(ACD) = AC + CD - AD$$

$$d(BEF) = BE + EF - BF$$

$$d(BFC) = BC + CF - BF.$$

One can use algebra to find the right chain from these data, and the answer is

$$ABC + ACD + BEF - BFC$$

CHECK THIS!

The boundary of this chain is

$$AB + BE + EF - CF + CD - AD.$$