SIMPLY CONNECTED REGIONS IN THE PLANE

Throughout this discussion we shall view the sphere S^2 as $\mathbf{R}^2 \cup \{\infty\}$, and we may refer to it as the *extended complex plane*.

Although the standard definition of simple connectivity involves the triviality of the fundamental group, an alternate characterization for open subsets of $\mathbf{R}^2$ is used in L. Ahlfors' classic text *Complex Analysis*. The purpose of this file is to explain why these notions are equivalent for connected open subsets of the complex plane and to note that the characterization does not extend to connected open subsets of $\mathbf{R}^n$ if $n \geq 3$.

Definition. A connected open subset U of the plane $\mathbf{R}^2$ is said to be *simply connected in the sense of Ahlfors' book* if and only if its complement $S^2 - U$ in the extended plane is connected. — This is the definition which appears in Section 4.4.2 of Ahlfors' book.

We then have the following result:

THEOREM. *Let U be an open connected subset of $\mathbf{R}^2$. Then U is simply connected in the usual sense if and only if it is simply connected in the sense of Ahlfors' book, in which case U is homeomorphic to $\mathbf{R}^2$.*

EXAMPLE. To see that this fails for $n \geq 3$, consider the open set

$$W \quad = \quad \left(\mathbf{R}^2 - \{\mathbf{0}\}\right) \;\times\; \mathbf{R}^{n-2} \quad \subset \quad \mathbf{R}^n \;.$$

Clearly $\mathbf{R}^2 - \{\mathbf{0}\}$ is a strong deformation retract of this open set, and therefore the fundamental groups of both are infinite cyclic. On the other hand, the complement of this set in $\mathbf{R}^n$ is homeomorphic to $\mathbf{R}^{n-2}$, so that the complement of W in the one point compactification S^n will be homeomorphic to the (connected) one point compactification of $\mathbf{R}^{n-2}$.

The proof of the theorem has two distinct parts.

Suppose that U is simply connected in the sense of Ahlfors' book. In Section 4.2 of his book, Ahlfors shows that his condition suffices for proving that line integrals of analytic functions along piecewise smooth curves are independent of path, and because of this one can prove results like Cauchy's Formula for regions satisfying his condition and the existence of an inverse to the exponential function on such an open set (in his notation, a branch of the function $\log z$). These results in turn are all that is needed to prove the Riemann Mapping Theorem, which states that if U is a connected proper subset of $\mathbf{R}^2$ which is simply connected in the sense of Ahlfors, then there is a 1–1 onto analytic map from U to the unit disk $N_1(\mathbf{0}) \subset \mathbf{R}^2$ which has an analytic inverse. The proof appears in Section 6.1 of the book, and it shows that if U is simply connected in the sense of Ahlfors' book then in fact U is homeomorphic to $\mathbf{R}^2$ (since $N_1(\mathbf{0})$ is homeomorphic to $\mathbf{R}^2$).■

Suppose now that U is simply connected in the usual sense. We shall prove this result using the Alexander Duality Theorem. One version of this theorem is stated in Section 3.3 of Hatcher (specifically, Theorem 3.44 on page 254), but we shall need a version which holds for arbitrary compact subsets of S^n. In this generality the result is given as Theorem 74.1 on page 445 of the following book:

J. R. Munkres. *Elements of Algebraic Topology*. Addison-Wesley, Reading, MA, 1984.

In order to state this result, we need the notion of *reduced zero-dimensional Čech cohomology*

$$\widetilde{H^0_{\check{C}}}(X)$$

for a topological space X. The fastest way to define this is to take the Cartesian product of an indexed family of infinite cyclic groups G_α, where α runs over all the quasi-components of X (see Munkres, *Topology*, pp. 163, 236), choose generators $g_\alpha \in G_\alpha$, and factor out the subgroup generated by the element whose coordinates are given by the classes g_α. Since a topological space is connected if and only if it has one quasi-component, it follows that X is connected if and only if its reduced zero-dimensional Čech cohomology is trivial.

If we restrict to subsets of S^2, this result can be restated as follows: *If K is a compact, locally contractible, nonempty, proper subspace of S^2 and $S^2 - K$ is connected, then the abelianization $H_1(S^2 - K)$ of the fundamental group $\pi_1(S^2 - K, \text{base point})$ is isomorphic to the reduced zero-dimensional Čech cohomology of K.*

NOTE. This statement uses the relationship between the 1-dimensional singular homology of an arcwise connected topological space and its fundamental group which is established in Section 2A of Hatcher; namely, the homology group is merely the abelianization of the fundamental group.

The strong version of Alexander Duality and the description of reduced zero-dimensional Čech cohomology yield the desired implication (ordinary simple connectivity implies simple connectivity in the sense of Ahlfors' book) very quickly. If U is simply connected and $K = S^2 - U$, then we know that the abelianization of $\pi_1(U)$ is trivial, and therefore by Alexander Duality it follows that the reduced zero-dimensional Čech cohomology of K is also trivial. But we have noted that the later holds if and only if K is connected, and therefore we see that $S^2 - U = K$ must be connected, which is what we wanted to show.■

Background on Čech cohomology

The following classic book is still one of the best references for Čech cohomology, and the statement about reduced zero-dimensional Čech cohomology is implicit in the exercises.

S. Eilenberg and N. Steenrod. *Foundations of Algebraic Topology*, Princeton Mathematical Series No. 15. Princeton University Press, Princeton, 1952.