Topological embeddings of graphs in graphs

The purpose of this document is to prove the following assertion which appears on page 362 of Munkres, *Topology*:

NONEMBEDDING EXAMPLE. *If E is a graph homeomorphic to a Figure Eight and T is a graph homeomorphic to a Figure Theta, then neither is homeomorphic to a subspace of the other.*

The proof of this assertion has two parts. One uses the local homology groups at points which are defined in Section VII.1 of `algtop-notes.pdf`, and the other involves an abstract result on topological embeddings of graphs which may be interesting for its own sake.

Throughout this document, a continuous mapping $f : X \to Y$ is called a (topological) *embedding* if it is a 1–1 continuous map which is a homeomorphism onto its image. By Theorem III.1.8 in `gentopnotes2014.pdf`, every continuous 1–1 mapping from a compact Hausdorff space into a Hausdorff space is a topological embedding, and by III.4.6 in the same notes every continuous strictly increasing mapping from an open interval (a, b) into the real line is a topological embedding with an open image. However, the Figure Eight curve $\varphi(t) = (\sin 2t, \sin t)$ for $t \in (0, 2\pi)$ is a regular smooth curve (differentiable with nowhere zero tangent vectors) which is 1–1, but its image is homeomorphic to a Figure 8 space (notice that the parametrization extends to $[0, 2\pi]$ and the extended curve passes through the origin at $t = 0$, π and 2π), so the image is compact but the domain is not.

We shall begin with the result on topological embeddings of one graph into another.

THEOREM 1. *Let $(X_0, \mathcal{E}_0)$ and $(X_1, \mathcal{E}_1)$ be finite graph complexes in the sense of* `algtop-notes.pdf`*, and let $h : X_0 \to X_1$ be a topological embedding. Then there are subdivisions $\mathcal{E}'_i$ of $\mathcal{E}_i$ (where $i = 0, 1$) such that h maps $(X_0, \mathcal{E}'_0)$ isomorphically onto a subgraph of $(X_1, \mathcal{E}'_1)$.*

One can informally summarize this result by saying that an embedding of one graph in another is well behaved or *tame*. When one passes to higher dimensions, the behavior of topological embeddings of graphs can be extremely wild, even for embeddings of S^1 into $\mathbb{R}^n$. The following textbook provides a comprehensive account of topological embeddings which exhibit various types of wild behavior.

> **T. B. Rushing,** Topological embeddings. Pure and Applied Mathematics, Vol. 52. *Academic Press, New York – London*, 1973. ISBN: 0-12-603550-4.

Proof of Theorem 1. Given a graph $(Y, \mathcal{G})$ and a finite set of points $V' \subset Y$, we can use the points in V' which are not vertices to construct a new subdivision whose vertices are the union of V' with the vertices of Y; specifically, if an edge E contains some nonvertex points in V', we use this points to split E into subintervals.

We shall now apply the reasoning of the preceding paragraph to $(X_0, \mathcal{E}_0)$ and $(X_1, \mathcal{E}_1)$ as follows: Let V_0 V_1 be the vertices in these respective graph complexes. Now add the vertices $h[V_0]$ to the vertices of $\mathcal{E}_1$ and let $(X_1, \mathcal{E}'_1)$ be the associated subdivision of $\mathcal{E}_1$ described above. Finally, add the vertices of $h^{-1}[V_1]$ and let $(X_0, \mathcal{E}'_0)$ be the associated subdivision of $\mathcal{E}_1$ described above.

We claim that $(X_0, \mathcal{E}'_0)$ is isomorphic to a subgraphof $(X_1, \mathcal{E}'_1)$. By construction the vertices of $(X_0, \mathcal{E}'_0)$ are exactly the vertices of $(X_1, \mathcal{E}'_1)$ which lie in X_0. Hence the proof of the assertion in the first sentence reduces to showing that if A is an edge of $(X_0, \mathcal{E}'_0)$, then $h[A]$ is an edge of $(X_1, \mathcal{E}'_1)$. Given the edge A of $(X_0, \mathcal{E}'_0)$, let $A^{\mathbf{O}}$ be the open edge obtained by removing the end points. Then the complement of $A^{\mathbf{O}}$ is a finite union of edges and isolated points, which implies the complement is closed and $A^{\mathbf{O}}$ is open. Since its image in X_1 does not contain any vertices of X_1, it follows that this image is contained in the union W_1 of the open edges in $(X_1, \mathcal{E}'_1)$. By construction the connected compoments of W_1 are the open edges of the latter graph, and since $A^{\mathbf{O}}$ is connected it follows that its image in X_1 must be contained in a single open edge; denote this open edge by $C^{\mathbf{O}}$, and let $C \in \mathcal{E}'_1$ be its closure. The continuity of the embedding h implies that A must be contained in C, and the condition on vertices implies that $C^{\mathbf{O}}$ does not contain either endpoint of A; if we combine these observations, we see that the endpoints of A must also be the endpoints of C. Since $h[A] \subset C$ and by continuity $h[A]$ must be a closed interval, it follows that $h[A] = C$, which is what we wanted to prove.■

Suppose now that we have an embedding h which maps the graph $(X_0, \mathcal{E}_0)$ isomorphically onto a subgraph of $(X_1, \mathcal{E}_1)$, and let v be a vertex of the subgraph. We shall also need the following elementary comparison principle relating the local homology groups at v in X_0 to the local homology groups at v in X_1.

PROPOSITION 2. *Suppose that $(X_0, \mathcal{E}_0)$ is a subgraph of $(X_1, \mathcal{E}_1)$ and v is a vertex of the subgraph. Then the local homology group $H_1(X_0, X_0 - \{v\})$ is isomorphic to a subgroup of $H_1(X_1, X_1 - \{v\})$.*

Proof. For $i = 0, 1$ let m_i be the number of edges in $(X_i, \mathcal{E}_i)$ which contain v as one of their endpoints. By Theorem VII.1.6 in `algtop-notes.pdf` we know that $H_1(X_i, X_i - \{v\})$ is free abelian of rank $m_i - 1$. Since X_0 is a subgraph of X_1 we know that $m_0 \leq m_1$ (every edge in the subgraph is also an edge in the larger graph), and if we combine this with the previous sentence we obtain the conclusion of the proposition.■

Proof of the assertion on page 362 of Munkres. Let E be the Figure Eight graph, and let T be the Figure Theta graph. The proof that neither is homeomorphic to a subset of the other splits naturally into two parts.

(1) *T is not homeomorphic to a subset of E.*

Suppose that there is a topological embedding $h : T \to E$. Then by Theorem 1 we can subdivide standard graph structures on T and E to realize some subdivision of T as a subgraph of some subdivision of E. Since there are two vertices v_1, v_2 of T such that $m_T(v_i) = 3$ and hence $H_1(T, T - \{v_i\}) \cong \mathbb{Z}^2$, we can use Proposition 2 to conclude that $H_1(E, E - \{v_i\}) \supset \mathbb{Z}^2$. However, there is only one vertex w of E such that $H_1(E, E - \{w\}) \cong \mathbb{Z}^m$ with $m \geq 2$, and hence we have a contradiction. The source of this contradiction is the assumption that there was a topological embedding of T in E, and consequently no such topological embedding can exist.

(2) *E is not homeomorphic to a subset of T.*

Suppose that there is a topological embedding $h : E \to T$. Then by Theorem 1 we can subdivide standard graph structures on E and T to realize some subdivision of E as a subgraph of some subdivision of T. Since there is one vertex w of E such that $m_E(w) = 4$ and hence $H_1(E, E - \{w\}) \cong \mathbb{Z}^3$, we can use Proposition 2 to conclude that $H_1(T, T - \{w\}) \supset \mathbb{Z}^3$. However, for every vertex v of T we have $H_1(T, T - \{v\}) \cong \mathbb{Z}^m$ with $m \leq 2$, and this yields a contradiction. The source of this contradiction is the assumption that there was a topological embedding of E in T, and consequently no such topological embedding can exist.■

Clearly the same methods are applicable to analyze many other pairs of examples.