Mathematics 205B, Spring 2019, Examination 1

Answer Key

1. [*25 points*] Let X be an arcwise connected space such that $X = U_1 \cup U_2$, where each U_i is open and arcwise connected and $U_1 \cap U_2$ is also arcwise connected. Denote the inclusions of the intersection into the U_i by q_i. Assume further that $\pi_1(U_1 \cap U_2) \cong \mathbb{Z} \times \mathbb{Z}$, $\pi_1(U_i) \cong \mathbb{Z}$ (both i), and the induced homomorphisms $q_{i*} : \pi_1(U_1 \cap U_2) \to \pi_1(U_i)$ correspond to the coordinate projections from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Prove that $\pi_1(X)$ is trivial.

SOLUTION

By the Seifert-van Kampen Theorem the fundamental group of X is given by the quotient of a free group on two generators (from $\pi_1(U_2) \cong \mathbb{Z}$ and $\pi_1(U_2) \cong \mathbb{Z}$) by the normal subgroup (normally) generated by all elements of the form $p_1(a,b) \cdot p_2(a,b)^{-1}$, where $p_1, p_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ are the coordinate projections.

Let's see what happens to the generators of $\mathbb{Z} \times \mathbb{Z}$. The map p_1 sends $(1,0)$ to the generator of $\pi_1(U_1)$, and the map p_2 sends it to the trivial element of $\pi_1(U_2)$. Therefore the generator of $\pi_1(U_1)$ lies in the normal subgroup that we are factoring out. Interchanging the roles of the two open sets and the coordinates, we see that the generator of $\pi_1(U_2)$ also lies in the normal subgroup that we are factoring out. Since these two generators are the generators of the free product $\pi_1(U_1) * \pi_1(U_2)$, it follows that the normal subgroup we are factoring out is equal to the entire free product, and therefore $\pi_1(X)$ must be trivial.■

NOTE. This situation is realized by a simple example: The sphere S^3 is a union of two open subsets U_1 and U_2 such that each is homeomorphic to $S^1 \times \mathbb{R}^3$, the intersection corresponds to $T^2 \times \mathbb{R}$, and the induced maps of fundamental groups are given as in the problem.■

2. [*25 points*] One of the following statements is true and the other one is false. State which one is true and give reasons for your answer. There is no need to give a counterexample to the false statement.

(1) If $U \subset \mathbb{R}^n$ is a nonempty open and arcwise connected subset, then U has a simply connected covering space.

(2) If $F \subset \mathbb{R}^n$ is a nonempty closed and arcwise connected subset, then F has a simply connected covering space.

SOLUTION

The first statement is true. To verify this, observe that if $U \subset \mathbb{R}^n$ and $x \in U$, then x has a neighborhood base of convex (hence simply connected) open sets; namely, the open disks $N_r(x)$, where r ranges over all sufficiently small real numbers (specifically, such that $N_r(x) \subset U$).■

Although the problem does not ask for a counterexample to the second statement, for the sake of completeness we note that Example 1 on page 487 of Munkres (often called the *Hawaiian earring*) does not have a simply connected covering.■

3. [*25 points*] (a) Let $(X, \mathcal{E})$ be a connected graph, and let $Y \subset X$ be a connected nonempty subgraph. Prove that if X is a tree, then Y is also a tree.

(b) Let $(X, \mathcal{E})$ be a connected graph with V vertices. Prove that the number E of edges is less than or equal to $f(V)$ for some quadratic polynomial f, and give a specific example of such a polynomial. Recall that an edge is determined by its endpoints.

SOLUTION

(a) Let y_1 and y_2 be vertices in Y. Since Y is connected, there is a reduced edge path in Y joining these vertices. This edge path is also a reduced edge path in the larger graph X. Since X is a tree, this is the only reduced edge path in X joining the vertices, and since there is only one such path in X there cannot be a second reduced edge path in Y joining y_1 and y_2.■

(b) Let $\mathcal{V}_2$ be the set of subsets in X with two elements, and let $\mathcal{E}$ be the set of edges. Then there is a 1–1 mapping $\mathcal{E} \to \mathcal{V}_2$ defined by taking an edge to the set of its end points. Hence the number of edges is no greater than the number of subsets in $\mathcal{V}_2$. Since there are V vertices, there are $\frac{1}{2}V(V-1)$ elements in $\mathcal{V}_2$, and consequently we must have $E \leq \frac{1}{2}V(V-1)$.■

4. [*25 points*] Let $X \subset \mathbb{R}^2$ be the union of the two closed curves defined by the equations $|x| + |y| = 3$ and $\max\{|x|, |y|\} = 2$. One can then define a connected graph complex structure on X which is symmetric with respect to both coordinate axes. Find all the vertices of one such structure which lie in the first quadrant ($x, y \geq 0$), and find the positive integer k such that $\pi_1(X)$ is a free group on k generators.

SOLUTION

Note first that this graph is symmetric with respect to both coordinate axes.

The vertices are given by the vertices of the two squares in the first sentence of the problem plus the points where the two squares meet. In the first quadrant these are points such that $x + y = 3$ and either $x = 2$ or $y = 2$. The vertices which satisfy these conditions are $(0, 3)$ and $(3, 0)$ for the square defined by the first equation, $(2, 2)$ for the second, and $(2, 1)$ and $(1, 2)$ for the intersection points.

In the first quadrant, there are three edges of the graph in the square defined by the first equation, so there are 12 edges coming from this square. Furthermore, each edge of the second square is split into three edges, so there are also 12 edges coming from the second square. Now there are 5 vertices in the first quadrant. However, the total number of vertices in the entire plane is equal to 16 because the each of the vertices $(0, \pm 3)$ and $(\pm 3, 0)$ lies in exactly two quadrants (for example, $(0, 3)$ lies in the first and second quadrants, while $(3, 0)$ lies in the first and fourth quadrants). Therefore the Euler characteristic χ of the graph is equal to $16 - 24 = -8$ and hence the fundamental group is free on $1 - \chi = 9$ generators.∎

There is a sketch of X on the next page.