1. [*30 points*] (a) Let $f : (X, x_0) \to (Y, y_0)$ be a basepoint preserving continuous mapping of connected spaces, and let $p : (E, e_0) \to (X, x_0)$ and $q : (T, t_0) \to (Y, y_0)$ be simply connected covering space projections. Prove that there is a unique continuous mapping $F : (E, e_0) \to (T, t_0)$ such that $q \circ F = f \circ p$.

(b) If f is a homeomorphism, prove that F is also a homeomorphism. [*Hint:* Let $g = f^{-1}$, and let G be constructed as in the first part. Why is $p \circ G \circ F = p = p \circ \mathrm{id}_E$ and what does this imply?]

SOLUTION.

(a) The Lifting Criterion states that such a map exists if and only if the image of $\pi_1(E)$ in $\pi_1(Y)$ under the map $(p \circ f)_* = p_* \circ f_*$ is contained in the image of q_*. Since the domain of $(p \circ f)_*$ is the trivial group, this condition is automatically satisfied.

(b) Let G be the lifting of $q \circ g$ which exists by (a). Then we have $q \circ F = f \circ p$ and $p \circ G = g \circ q$. Therefore

$$p \circ G \circ F \quad = \quad g \circ q \circ F \quad = \quad g \circ f \circ p \quad = \quad p$$

where the last equation follows from the assumption that $g = f^{-1}$. Since $p = p \circ \mathrm{id}_E$, this yields the equations in the last sentence of the hint. By construction, both $G \circ F$ and the identity on E are liftings of $p : E \to B$ to E which preserve basepoints, and therefore the maps $G \circ F$ and id_E are equal by the uniqueness of liftings. If we interchange the roles of $\{p, E, X, q, T, Y, F, G\}$ with those of $\{q, T, Y, p, E, X, G, F\}$ in the preceding argument, we also obtain the identity $F \circ G = \mathrm{id}_T$. Therefore F is a homeomorphism and G is its inverse.

2. [*25 points*] Suppose that the arcwise connected space is the union of the arcwise connected open subsets U and V such that $U \cap V$ is also arcwise connected, and assume further that the inclusion induced homomorphism $\pi_1(U \cap V) \to \pi_1(V)$ is an isomorphism. Prove that the inclusion induced homomorphism $\pi_1(U) \to \pi_1(X)$ is also an isomorphism. [*Hint:* If $\alpha : K \to H$ is a group isomorphism and $\beta : K \to G$ is a homomorphism, show that

$$\begin{array}{ccc} K & \xrightarrow[\cong]{\alpha} & H \\ \downarrow \beta & & \downarrow \beta\alpha^{-1} \\ G & \xrightarrow{=} & G \end{array}$$

has the Universal Mapping Property for pushouts.]

SOLUTION.

Since the fundamental group of X is the pushout of the diagram

$$\pi_1(V) \leftarrow \pi_1(U \cap V) \rightarrow \pi_1(U)$$

it will suffice to prove that the diagram in the hint has the Universal Mapping Property for pushouts.

Now suppose that we have homomorphisms f, g in a commutative diagram as below, where $K = \pi_1(U \cap V)$, $G = \pi_1(U)$, and $H = \pi_1(V)$; the homomorphisms α and β are taken to be induced by the appropriate inclusion mappings:

$$\begin{array}{ccc} K & \xrightarrow[\cong]{\alpha} & H \\ \downarrow \beta & & \downarrow g \\ G & \xrightarrow{f} & \Gamma \end{array}$$

We need to show that there is a unique homomorphism $\theta : G \to \Gamma$ such that $\theta \circ \mathrm{id}_G = f$ and $\theta \circ (\beta\alpha^{-1}) = g$.

Uniqueness. The first condition implies that if θ has the desired properties then $\theta = \theta \circ \mathrm{id}_G = f$.

Existence. We claim that f has the desired properties. The first follows from $\theta \circ \mathrm{id}_G = f$, and if we compose the relation $f \circ \beta = g \circ \alpha$ with α^{-1} we see that $f \circ (\beta\alpha^{-1}) = g$.

3. [*20 points*] Let X be a connected space whose fundamental group is cyclic of order 30. Describe a finite set $\mathcal{M}$ of covering space projections for X such that every connected covering of X is equivalent to an example in $\mathcal{M}$. Assume also that X is locally simply connected.

SOLUTION.

The connected covering spaces are classified by the the subgroups of $\pi_1(X) = \mathbb{Z}_{30}$, and the associated subgroup is just the fundamental group of the covering space. Therefore we need only count the subgroups of $\mathbb{Z}_{30}$.

These subgroups are given by the finite cyclic groups whose orders divide 30 (we include the trivial group as a cyclic group of order 1). Now $30 = 2 \cdot 3 \cdot 5$, so the subgroups are given by the eight integers $d = 2^\alpha 3^\beta 5^\gamma$ where $\alpha, \beta, \gamma \in \{0, 1\}$. Let $\widetilde{X}$ be the universal covering space of X, and for each such d let X_d be the covering space $\widetilde{X}/\mathbb{Z}_d$. Then the fundamental group of X_d is cyclic of order d and hence every covering space is equivalent to one of the eight coverings whose total spaces are given by X_d where d runs through the eight integral divisors of 30.

4. [*25 points*] If X is the union of the edges in a triangular prism, then X is decomposed into 9 edges with a total of 6 vertices a, b, c, d, e, f and edges given by the following list:

$$ab, bc, ac, de, ef, df, ad, be, cf$$

Find a maximal tree in X with respect to this decomposition and compute the fundamental group of X. It will suffice to describe the maximal tree by means of an accurate sketch. [*Hint:* The graph can be embedded in the plane with $\triangle def$ sitting inside $\triangle abc$ such that the lines ab, bc, ac are parallel to de, ef, df respectively.]

SOLUTION.

We can compute the fundamental group by computing the Euler characteristic χ, which is $v - e = 6 - 9 = -3$. The number of generators is then equal to $1 - (-3) = 4$.

There are many different possibilities for the maximal tree, and they are precisely the subgraphs T satisfying the following three conditions:

(1) T is connected.

(2) T contains all the vertices of X.

(3) T contains exactly five edges of X.

Finding all the subgraphs satisfying these conditions might be an interesting or useful exercise; in fact, one can write a reasonably short computer program to find all the possibilities. There are

$$\binom{9}{5} \quad = \quad 126$$

subgraphs with exactly five edges, and one can test each one to see if (1) and (2) are satisfied.