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Mathematics 205B, Winter 2018, Examination 2

Answer Key

Recall: All spaces are assumed to be Hausdorff and locally arcwise connected, and we also assume the existence of a singular homology theory with the properties specified in the course.

1. [*25 points*] Prove that there is no continuous 1–1 mapping $f : S^m \to S^n$ if $m > n$. [*Hint:* Let $j : S^n \to S^m$ denote the standard inclusion and consider the mapping $j \circ f : S^m \to S^m$. Why is it 1–1, and why is the image not open?]

SOLUTION

Follow the hint. Since the composite of continuous 1–1 mappings is continuous and 1–1, it follows that $j \circ f : S^m \to S^m$ is continuous and 1–1. The image of this map is a set $A \subset S^n$. By Invariance of Domain it follows that A must be open.

But no subset of S^n can be open. By construction, all points in $S^n \subset S^m$ have a zero in the last coordinate. However, every open neighborhood of such a point must contain points with both positive and negative last coordinates, so A fails to be open. This contradicts the conclusion in the preceding paragraph. The source of the contradiction is the assumption that a mapping like f exists, and therefore no such mapping can exist.■

Alternative argument. Take the reverse composite $g = f \circ j : S^n \to S^n$. Once again this is a 1–1 continuous mapping, so that it is open by Invariance of Domain. We claim g is a homeomorphism; in other words, g must be onto.

By Invariance of Domain we know that $g[S^n]$ is open, and since S^n is compact we also know that $g[S^n]$ is compact, hence closed, in S^n. By connectedness, it follows that g is onto. If we now define $\rho : S^m \to S^n$ to be $g^{-1} \circ f$, it follows that $\rho \circ j$ is the identity on S^n and hence S^n is a retract of S^m. This is false, for if it were true then $H_n(S^n) \cong \mathbb{Z}$ would be isomorphic to a subgroup of $H_n(S^m) = 0$. As in the preceding argument, the source of the contradiction is the assumption that a mapping like f exists, and therefore no such mapping can exist.■

2. [*25 points*] Let (X, a) and (Y, b) be pointed connected spaces, and let $f : (X, a) \to (Y, b)$ be continuous and basepoint preserving. Suppose that we are given two basepoint preserving covering space projections $p_1 : (X^*, a^*) \to (X, a)$ and $p_2 : (Y^*, b^*) \to (Y, b)$ such that X^* and Y^* are simply connected. Prove that there is a unique continuous mapping $F : (X^*, a^*) \to (Y^*, b^*)$ such that $p_2 \circ F = f \circ p_1$. Also, prove that if f is a homeomorphism then so is F.

SOLUTION

For the first part, consider the mapping $f \circ p_1$; this lifts to a mapping F of the required type if and only if the image of $\pi_1(X^*)$ in $\pi_1(Y)$ under $(f \circ p_1)_*$ is contained in the image of p_{2*}. Since both of these images are trivial, the sufficient condition in this Lifting Criterion is satisfied and hence the desired lifting exists. Furthermore, the uniqueness part of the theorem on liftings implies there is a uniqe F with the required properties.

To see the statement about homeomorphisms, let $g = f^{-1}$ and construct G as in the first part. It suffices to show that $F \circ G$ and $G \circ F$ are both identity mappings. We know both are basepoint preserving, with the first lifting id_Y and the second lifting id_X. By the uniqueness part of the theorem on liftings, it follows that $F \circ G$ is the identity on Y^* and $G \circ F$ is the identity on X^*.■

3. [*30 points*] Let $\mathbf{K}$ be an n-dimensional simplicial complex (with underlying space P and a linear ordering ω of the vertices), and let $\mathbf{K}_0$ be a subcomplex of $\mathbf{K}$ which contains all the simplices of $\mathbf{K}$ except for one n-simplex. Furthermore, assume that $H_n(\mathbf{K}) \cong \mathbb{Z}$ with a generator of the form $\sum \varepsilon_j \, \sigma_j$, where σ_j runs through the n-simplices of $\mathbf{K}$ and each ε_j is ± 1.

(a) Using simplicial excision, explain why the map $H_n(\mathbf{K}) \to H_n(\mathbf{K}, \mathbf{K}_0)$ is an isomorphism.

(b) Why does (a) imply that $H_n(\mathbf{K}_0) = 0$? [*Hint:* What can we say about the group $H_{n+1}(\mathbf{K}, \mathbf{K}_0)$?]

SOLUTION

(a) Look on the chain level, and consider the image of $\sum \varepsilon_j \, \sigma_j$ in $C_n(\mathbf{K}, \mathbf{K}_0)$. Since $\mathbf{K}_0$ contains all the simplices of $\mathbf{K}$ except for one n-simplex the image in the relative chain complex must be $\varepsilon_A \, \sigma_A$, where A is the unique n-simplex not in $\mathbf{K}_0$. By excision the maps $C_n(A, \mathbf{K}_0 \cap A) \to C_n(\mathbf{K}, \mathbf{K}_0)$ and $H_n(A, \mathbf{K}_0 \cap A) \to H_n(\mathbf{K}, \mathbf{K}_0)$ are isomorphisms; since $\mathbf{K}_0$ consists of all simplices in $\mathbf{K}$ except A, it follows that $\mathbf{K}_0 \cap A$ is the boundary ∂A, and hence the simplicial chains and homology of the pair $(A, \mathbf{K}_0 \cap A) = (A, \partial A)$ are $\mathbb{Z}$ in dimension n and 0 otherwise. Therefore $\varepsilon_A \, \sigma_A$ represents a cycle generating $H_n(A, \partial A) = H_n(A, \mathbf{K}_0 \cap A)$, and it follows that this chain represents a generator for $H_n(\mathbf{K}, \mathbf{K}_0)$. Since this class represents the image of a generator for $H_n(\mathbf{K})$ under the map $j_* : H_n(\mathbf{K}) \to H_n(\mathbf{K}, \mathbf{K}_0)$, it follows that j_* takes a generator to a generator and hence is an isomorphism.■

(b) To answer the question in the hint, we know that $H_{n+1}(\mathbf{K}, \mathbf{K}_0) = 0$ because $\mathbf{K}$ is an n-dimensional complex and hence has no nontrivial chains in higher dimensions. Therefore we have an exact sequence

$$0 = H_{n+1}(\mathbf{K}, \mathbf{K}_0) \to H_n(\mathbf{K}_0) \to H_n(\mathbf{K}) \to H_n(\mathbf{K}, \mathbf{K}_0)$$

in which the last map is an isomorphism. Therefore the map $H_n(\mathbf{K}_0) \to H_n(\mathbf{K})$ must be trivial, which means that $0 = H_{n+1}(\mathbf{K}, \mathbf{K}_0) \to H_n(\mathbf{K}_0)$ must be onto. It follows immediately that $H_n(\mathbf{K}_0) = 0$.■

4. [*20 points*] Let U and V be convex open subsets of $\mathbb{R}^n$ for some n, *and assume that* $U \cap V$ *is nonempty*. Prove that $H_q(U \cup V) = 0$ if $q \geq 1$. [*Hint:* What do we know about the intersection of two convex sets?]

SOLUTION

This can be done using the Mayer-Vietoris for $U \cup V$ in reduced homology:

$$\widetilde{H}_q(U \cap V) \to \widetilde{H}_q(U) \oplus \widetilde{H}_q(V) \to \widetilde{H}_q(U \cup V) \to \widetilde{H}_{q-1}(U \cap V)$$

The sets U, V and intersection $U \cap V$ are nonempty convex sets, and hence they are contractible; it follows that the reduced homology groups yield a short exact sequence of the form

$$0 \to 0 \oplus 0 \to \widetilde{H}_q(U \cup V) \to 0 \ .$$

By exactness we must also have $\widetilde{H}_q(U \cup V) = 0$ for all q. The conclusion follows because $H_q = \widetilde{H}_q$ if $q > 0$.■

Alternative argument. If $p \in U \cap V$, then $U \cap V$ is starshaped with respect to p. Therefore $U \cup V$ is contractible and hence $H_q(U \cup V) = H_q(\{p\}) = 0$ if $q \geq 1$.■

5. [*25 points*] Suppose that X is a contractible space and $A \subset X$ is a subspace. Prove that $H_{q+1}(X, A)$ is isomorphic to $H_q(A)$ for all $q > 0$.

SOLUTION

Since X is contractible we know that $H_q(X) = 0$ for all $q > 0$. Therefore the long exact homology sequence for the pairs breaks up into short segments

$$0 = H_{q+1}(X) \longrightarrow H_{q+1}(X, A) \longrightarrow H_q(A) \longrightarrow H_q(X) = 0$$

for each $q > 0$ (note that this fails when $q = 0$). Whenever we have an exact sequence of the form $0 \to A \to B \to 0$, the map $A \to B$ is an isomorphism (the kernel is the image of $0 \to A$, and the image is the kernel of $B \to 0$), and therefore it follows that $H_{q+1}(X, A) \to H_q(A)$ is an isomorphism for all $q > 0$.■

6. [*25 points*] Let X be a nonempty topological space, and let $j : X \to X \times [0,1]$ send x to $(x,0)$. Prove that the induced homomorphism j_* in singular homology is an isomorphism in each dimension. [*Hint:* Consider the coordinate projection $p : X \times [0,1] \to X$ and the composites $p \circ j$, $j \circ p$.]

SOLUTION

We shall show that $j[X] = X \times \{0\}$ is a strong deformation retract of they cylinder $X \times [0,1]$.

Follow the hint. By construction we have $p \circ j(x) = p(x,0) = x$, so that $p \circ j = \mathrm{id}_X$, and $j \circ p(x,t) = (x,0)$. If $H(x,t;s) = (x, s + (1-s)t)$, then $H(x,t,0) = (x,t)$, $H(x,t,1) = (x,0)$ and $H(x,0,s) = (x,0)$, and hence $j \circ p$ is homotopic to the identity on $X \times [0,1]$ by a homotopy which is fixed on $X \times \{0\}$. In particular, this means that j is a homotopy equivalence, and by the homotopy invariance property of singular homology it follows that the induced mappings j_* in homology are isomorphisms.■