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Mathematics 145B, Spring 2017, Examination 3

Answer Key

1. [*25 points*] Let $(X, \mathcal{E})$ be a graph, and let $E_1 \cdots E_r$ be an edge path (not necessarily reduced!) in X. Prove that the union of the edges E_i is a connected subset of X.

SOLUTION

One way to do this is to prove by induction that $A_k = \cup_{i \leq k} E_i$ is connected for each k such that $1 \leq k \leq r$. If $k = 1$ this follows because E_1 is homeomorphic to the closed interval $[0, 1]$ and the latter is connected. Suppose now that A_k is connected, where $1 \leq k < r$; if we can prove that the latter implies A_{k+1} is connected, then the result will follow by induction.

By construction $A_{k+1} = A_k \cup E_{k+1}$, so A_{k+1} is a union of two connected subsets. The intersection of these subsets contains the vertex v_k and hence is nonempty. Therefore A_{k+1} is also connected by a proposition on unions of connected subsets (which have a nonempty intersection).■

2. [*25 points*] Let $g : S^1 \to \mathbb{R}^2 - \{\mathbf{0}\}$ be the circle with radius 1 and center $\frac{1}{2}$ given in complex numbers by $g(z) = z + \frac{1}{2}$, and let $f(z) = z$ define the usual circle. Prove that f and g are homotopic as mappings from S^1 to $\mathbb{R}^2 - \{\mathbf{0}\}$. [*Note and Hints:* Since $f(z) \neq g(z)$ for all z, these maps are not base point preservingly homotopic. What is the simplest method for trying to define a homotopy between two maps into an open subset of $\mathbb{R}^n$? It may be very helpful to draw a picture of the images of f and g.]

SOLUTION

See the file `exam3s17-fig1.pdf` for a drawing. Let H be the straight line homotopy $H(z,t) = (1-t)f(z) + tg(z) = (1-t)z + t\left(z + \frac{1}{2}\right)$. If we can show that $H(z,t)$ is never zero, then H will define a homotopy from f to g as mappings from S^1 to $\mathbb{R}^2 - \{\mathbf{0}\}$. Now the expression for $H(z,t)$ simplifies to $z + t\,\frac{1}{2}$, so we need to show that this is never zero if $z \in S^1$. This is clear from the drawing, and one can prove it formally as follows: Suppose that we write $z = x + yi$, so that $H(z,t) = \left(x + t\,\frac{1}{2}\right) + yi$. This is zero if and only if the real and imaginary parts are zero, so it is zero if and only if $y = 0$ and $x = -t\,\frac{1}{2}$. In particular, since $0 \leq t \leq 1$ it follows that $|z| = |x| \leq \frac{1}{2}$. Therefore if $|z| = 1$ the value of $H(z,t)$ is nonzero, and therefore H defines a homotopy of mappings into $\mathbb{R}^2 - \{\mathbf{0}\}$.■

3. [*25 points*] Suppose that $f : X \to Y$ is a homotopy equivalence, and $g, h : Y \to X$ are homotopy inverses to f. Prove that g and h are homotopic, and give an example of two distinct homotopy inverses for the identity mapping on the closed interval $[0,1]$. [*Hint:* For the first part, analyze the homotopy class of the composite map $h \circ f \circ g : Y \to X$.]

SOLUTION

For the first part, follow the hint and consider the homotopy class of $h \circ f \circ g : Y \to X$. Since $f \circ g$ is homotopic to the identity on Y, it follows that $h \circ f \circ g \simeq h \circ \mathrm{id}_Y = h$. On the other hand, since $g \circ f$ is homotopic to the identity on X, it follows that $h \circ f \circ g \simeq \mathrm{id}_X \circ g = g$. Therefore both g and h are homotopic to of $h \circ f \circ g$, and this implies that $h \simeq g$.■

For the second part, the identity on $[0,1]$ is a continuous inverse to itself and is thus a homotopy inverse, and the constant map sending everything to 0 is another homotopy inverse.■

4. [*25 points*] Let (X, d) be a discrete metric space with the standard metric such that $d(u, v) = 1$ if $u \neq v$. Prove that (X, d) is complete. [*Hint:* Why is every Cauchy sequence eventually constant?]

SOLUTION

Let $\{a_n\}$ be a Cauchy sequence in X. Then by the definition of a Cauchy sequence there is some positive integer M such that $n, m \geq M$ implies $d(a_m, a_n) < 1$. However, by the construction of the metric we know that two points are equal if their distance is less than 1, so the preceding sentence implies that $a_n = a_M$ if $n \geq M$, for then $n \geq M$ implies $d(a_M, a_n) = 0$. In particular, for all $\varepsilon > 0$ we have that $n \geq M$ implies $d(a_M, a_n) < \varepsilon$. But this means that $a_M = \lim_{n\to\infty} a_n$, so that the Cauchy sequence converges to a_M.■

5. [*25 points*] For each of the following, state whether the assertion is always true, sometimes true and sometimes false, or always false. Correct answers will receive full credit, and incorrect answers with reasons may receive partial credit. — In each case, let $\mathcal{R}$ be an equivalence relation on the unit interval $\mathbf{I} := [0,1]$

(a) The quotient space $\mathbf{I}/\mathcal{R}$ is compact.

(b) The quotient space $\mathbf{I}/\mathcal{R}$ is connected.

(c) The quotient space $\mathbf{I}/\mathcal{R}$ is metrizable (the quotient topology comes from a metric).

(d) The space $\mathbf{I}$ is not homeomorphic to a quotient space of $\mathbf{I}/\mathcal{R}$. [*Hint:* Draw a circle in the plane which has $\mathbf{I}$ as a diameter.]

SOLUTION

(a) ALWAYS TRUE because the map $\mathbf{I} \to \mathbf{I}/\mathcal{R}$ is continuous and onto, and the continuous image of a compact space is compact.■

(b) ALWAYS TRUE because the map $\mathbf{I} \to \mathbf{I}/\mathcal{R}$ is continuous and onto, and the continuous image of a connected space is connected.■

(c) SOMETIMES TRUE AND SOMETIMES FALSE because (i) if $\mathcal{R}$ is the equivalence relation with $x \sim y \Leftrightarrow x = y$ then $\mathbf{I}/\mathcal{R}$ is homeomorphic to $\mathbf{I}$, (ii) if $\mathcal{R}$ is the equivalence relation whose equivalence classes are $[0, \frac{1}{2})$, $\{\frac{1}{2}\}$, and $(\frac{1}{2}, 1]$ then there are one point subsets of $\mathbf{I}/\mathcal{R}$ which are not closed (and the latter is true for metrizable spaces).■

(d) SOMETIMES TRUE AND SOMETIMES FALSE because (i) if $\mathcal{R}$ is the equivalence relation with $x \sim y \Leftrightarrow x = y$ then $\mathbf{I}/\mathcal{R}$ is homeomorphic to $\mathbf{I}$, (ii) if $\mathcal{R}$ is the equivalence relation whose equivalence classes are $[0, \frac{1}{2})$, $\{\frac{1}{2}\}$, and $(\frac{1}{2}, 1]$ then $\mathbf{I}/\mathcal{R}$ is finite and an infinite space like $\mathbf{I}$ cannot be a quotient space of a finite space (in fact, if X has k elements and $\mathcal{R}$ is an equivalence relation on X, then $X/\mathcal{R}$ has at most k elements).■

6. [*25 points*] Let (X, x_0) be a simply connected pointed space (arcwise connected + trivial fundamental group). Prove that (X, x_0) and $\left(X \times S^1, (x_0, 1)\right)$ are not base point preservingly homotopy equivalent. [*Hint:* Compare the fundamental groups of the two spaces.]

SOLUTION

If two arcwise connected spaces are base point preservingly homotopy equivalent, then the mapping properties of fundamental groups imply that the spaces' fundamental groups are algebraically isomorphic. Hence it suffices to prove that $\pi_1(X, x_0)$ and $\pi_1\left(X \times S^1, (x_0, 1)\right)$ are not isomorphic as groups.

Since X is simply connected we know that $\pi_1(X, x_0) = \{1\}$. By the product formula for fundamental groups we also know that

$$\pi_1\left(X \times S^1, (x_0, 1)\right) \quad \cong \quad \pi_1(X, x_0) \times \pi_1(S^1, 1) \quad \cong \quad \mathbb{Z}$$

is infinite. Therefore there is not even a 1–1 correspondence of sets between these fundamental groups, much less a group isomorphism, and it follows that the pointed spaces (X, x_0) and $\left(X \times S^1, (x_0, 1)\right)$ are not base point preservingly homotopy equivalent.■