EXERCISES FOR MATHEMATICS 205B

WINTER 2012

File Number 00

These are mainly review from the previous course in the sequence.

1. (i) Let U be an open subset in $\mathbb{R}^n$ for some n. Prove that U has countably many arc components. [*Hint:* Why is every point in the same arc component as a point with rational coordinates.

(ii) Given two homotopy equivalent spaces X and Y, prove that there is a 1–1 correspondence between their sets of arc components.

(iii) Let X be the Cantor Set constructed as $\cap_n X_n$, where $X_0 = [0,1]$, X_n is a union of 2^n pairwise disjoint closed intervals of length 3^{-n}, and X_{n+1} is obtained from X_n by removing the open middle third from each interval. Prove that X cannot have the homotopy type of an open subset in $\mathbb{R}^n$ for any n. [*Hint:* Given two points $u \neq v \in X$, find U and V be disjoint open subsets containing u and v such that $U \cup V = X$. Why does this imply that every arc component of X consists of a single point?]

2. Let Y be a nonempty topological space with the indiscrete topology (*i.e.*, $\emptyset$ and Y are the only open sets), and let X be an arbitrary nonempty topological space. Prove that $[X, Y]$ consists of a single point. [*Hint:* For all topological spaces W, every map of sets from W to Y is continuous. Using this, show that if $A \subset B$ is a subspace and $g : A \to Y$ is continuous, then g extends to a continuous map from B to Y.]

3. Let U be an open subset in $\mathbb{R}^n$ for some n, and let $u_0 \in U$ be a point with rational coefficients.

(i) Let K be a compact metric space, and let $f : K \to \mathbb{R}^n$ be continuous. Prove that there is some $\varepsilon > 0$ such that if the distance from $x \in \mathbb{R}^n$ to $f[K]$ is less than ε, then $x \in U$.

Definition. Let U be an open subset of $\mathbf{R}^n$. A **broken line curve** is a continuous curve $\gamma : [a, b] \to U$ such that the following holds: There is a partition of $[a, b]$ given by

$$a \;=\; x_0 \;<\; x_1 \;<\; \cdots \;<\; x_k \;=\; b$$

such that the restriction of γ to each closed subinterval $[x_{i-1}, x_i]$ is a straight line segment which has a parametrization of the form

$$\beta(t) \;=\; \left(\frac{x_{i-1} - t}{\Delta_i}\right) \cdot \gamma(x_{i-1}) \;+\; \left(\frac{t - x_i}{\Delta_i}\right) \cdot \gamma(x_i)$$

where $\Delta_i = x_i - x_{i-1}$. The points $\gamma(a)$ and $\gamma(b)$ are called the initial and final points, and the remaining points of the form $\gamma(x_i)$ are called corner points.

(*ii*) Let $\gamma : [0,1] \to U$ be a closed curve such that $\gamma(0) = \gamma(1) = u_0$. Prove that the class of γ is also represented by a broken line curve as above such that the initial and final points are $\gamma(0) = \gamma(1)$ and the corner points $\beta_i \in U$ have rational coefficients. [*Hint:* Let $\varepsilon > 0$ be as in the first point of the exercise, use uniform continuity to find some $\delta > 0$ such that $|t - s| < \delta$ implies $|\gamma(t) - \gamma(s)| < \frac{1}{4}\varepsilon$, and partition $[0,1]$ into subintervals of length less than δ. Suppose the partition is given by $0 = x_0 < x_1 < \cdots < x_k = 1$, and for $i = 1, ..., k-1$ choose β_i such that $|\beta_i - \gamma(x_i)| < \frac{1}{4}\varepsilon$, and show that the corresponding broken line is base point preservingly homotopic in U to the original curve γ. More precisely, show that the line segment joining β_i to β_{i+1} lies in the open ε-disk centered at $\gamma(x_i)$; we know that the latter is contained in U.]

(*iii*) Using the preceding, explain why $\pi_1(U, u_0)$ is countable. [*Hint:* Count the number of finite sequences of points in U with rational coordinates.]

4. Prove the **compact support property** of the fundamental group: If $\alpha \in \pi_1(X, x)$, then there is a compact subset K of X such that $x \in K$ and α lies in the image of the map $\pi_1(K, x) \to \pi_1(X, x)$. Furthermore, if α' and β' in $\pi_1(K, x)$ map to the same element of $\pi_1(X, x)$, then there is some compact set $L \subset X$ such that L contains K and α' and β' map to the same element of $\pi_1(L, x)$. [*Hint:* If we choose representative curves or homotopies, their images are compact.]

5. Let $p : E \to X$ be a covering map, and let $f : Y \to X$ be continuous. Define the *pullback*

$$Y \times_X E := \{(e, y) \in Y \times E |\ f(y) = p(e)\}.$$

Let $p_{(Y,f)} = \text{proj}_Y | Y \times_X E$.

(*i*) Prove that $p_{(Y,f)}$ is a covering map. Also prove that f lifts to E if and only if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)} s = 1_Y$.

(*ii*) Suppose also that f is the inclusion of a subspace. Prove that there is a homeomorphism $h : Y \times_X E \to p^{-1}(Y)$ such that $p \circ h = p_{(Y,f)}$.

NOTATION. If the condition in (*ii*) holds we sometimes denote the covering space over Y by $E|Y$ (in words, E restricted to Y).

6. (*i*) Suppose that $A \subset X$ is a retract and X is Hausdorff. Prove that A is a closed subset of X. [*Hint:* Let $i : A \subset X$ be the inclusion and let $r : X \to A$ be the retraction. What can we say about the set of all points $y \in X$ such that $i \circ r(y) = y$?]

(*ii*) Suppose we have $A \subset B \subset X$ such that A is a retract of B and B is a retract of X. Prove that A is a retract of X.

(*iii*) Suppose that A is a retract of X; let $j : A \to X$ be the inclusion mapping, and let $x_0 \in A$. If H is the image of the fundamental group of A under the mapping h_*, prove that there is a normal subgroup K of $\pi_1(X, x_0)$ such that the latter is generated by H and K, and we have $H \cap K = \{1\}$. [*Hint:* Let $r : X \to A$ be the associated retraction, and consider the kernel of r_*.]

7. Prove that S^n is simply connected for all $n \geq 2$. [*Hint:* Since S^n is a deformation retract of $\mathbb{R}^{n+1} - \{\mathbf{0}\}$, it suffices to prove this for $\mathbb{R}^{n+1} - \{\mathbf{0}\}$. By Exercise 00.3 it suffices to that if γ is a closed broken line curve which is based at e_1 and has corner points with rational coordinates, then γ is homotopically trivial. As usual, let the corner points be given by $\gamma(x_i)$, and let W_i be the proper vector subspace spanned by $\gamma(x_i)$ and γ_{i-1}; this subspace is proper because $n + 1 \geq 3$.

Why is there a unit vector $\mathbf{u}$ such that the span of $\mathbf{u}$ and each W_i have no nonzero vectors in common? Why does this imply that the image of γ is contained in $\mathbb{R}^{n+1} - \mathrm{Span}(\mathbf{u})$, why is the latter homeomorphic to $(S^n - \{\mathbf{u}\})$, and why is the this space homeomorphic to $\mathbb{R}^{n+1}$? Show that the preceding observations imply that γ must be homotopically trivial.]