

# EXERCISES FOR MATHEMATICS 205C

## SPRING 2011

File Number 04

DEFAULT HYPOTHESES. Unless specifically stated otherwise, all spaces are assumed to be Hausdorff and locally arcwise connected.

0. (a) Suppose that the 2-dimensional simplicial complex  $(P, \mathbf{K})$  has vertices  $A, B, C, D, E$  and simplices given by

$$ABC, ABD, ACD, BCD, BCE, BDE, CDE$$

and all their faces. Let  $\mathbf{K}_1$  and  $\mathbf{K}_2$  be the subcomplexes consisting of all simplices lie in the sets  $\{A, B, C, D\}$  and  $\{B, C, D, E\}$  respectively. Compute the homology of  $(P, \mathbf{K})$  using a Mayer-Vietoris sequence. [*Hint:* Each  $\mathbf{K}_i$  is isomorphic to the boundary of a 3-simplex, and their intersection is the simplex  $BCD$ .]

(b) Suppose that the 2-dimensional simplicial complex  $(P, \mathbf{K})$  has vertices  $A, B, C, D, X$  and simplices given by

$$ABC, ABD, ACD, BCD, AX, BX, CX, DX$$

and all their faces. Let  $\mathbf{K}_1$  be the subcomplex generated by the first four of these simplices and let  $\mathbf{K}_2$  be the subcomplex generated by the last four. Compute the homology of  $(P, \mathbf{K})$  using a Mayer-Vietoris sequence. [*Hint:* Every simplex in the second subcomplex is contained in a simplex whose vertices include  $X$ , and the intersection of the two subcomplexes is finite.]

(c) Compute the homology of  $(P, \mathbf{K})$  if  $\mathbf{K}$  is the 2-skeleton of the simplex  $\Delta_4$ ; recall that the  $m$ -skeleton of a simplicial complex is generated by all simplices of dimension  $\leq m$ . [*Hint:* Computing the relative homology of  $(\Delta_4, \mathbf{K})$  is fairly easy to do; note that it is nonzero in only two dimensions.]

1. Prove that the homotopy invariance axiom for homology theories is equivalent to the following weaker statement, which is supposed to hold for all pairs  $(X, A)$ :

If  $t = 0, 1$  and  $i_t : (X, A) \rightarrow (X \times [0, 1], A \times [0, 1])$  is the slice inclusion  $i_t(x) = (x, t)$ , then the maps in homology  $i_{0*}$  and  $i_{1*}$  are equal.

[*Hint:* If  $H$  is a homotopy from  $f_0$  to  $f_1$  then  $f_t = H \circ i_t$  for  $t = 0, 1$ .]

2. Given a topological space  $X$ , the **cone** on  $X$ , written  $\mathbf{C}(X)$ , is defined to be the quotient space of  $X \times [0, 1]$  whose equivalence classes are the one point subsets  $(x, t)$  for  $t > 0$  and the subset  $X \times \{0\}$ , which is called the *vertex* of the cone.

(a) If  $f : X \rightarrow Y$  is a continuous mapping, explain why the mapping  $f \times \text{id}_{[0,1]}$  from  $X \times [0, 1]$  to  $Y \times [0, 1]$  passes to a continuous map  $\mathbf{C}(f) : \mathbf{C}(X) \rightarrow \mathbf{C}(Y)$  and this construction has the functorial properties  $\mathbf{C}(g \circ f) = \mathbf{C}(g) \circ \mathbf{C}(f)$  and  $\mathbf{C}(\text{id}) = \text{id}$ .

(b) Prove that  $\mathbf{C}(X)$  is contractible [*Hint*: Show that the identity map is homotopic to the map which sends everything to the vertex.] Using this, show that  $H_q(\mathbf{C}(X), X \times \{1\})$  is isomorphic to  $H_{q-1}(X)$  if  $q \geq 2$ , and  $H_1(\mathbf{C}(X), X \times \{1\}) = 0$  if  $X$  is arcwise connected.

**3.** Given a topological space  $X$ , the **suspension** on  $X$ , written  $\Sigma(X)$ , is defined to be the quotient space of  $X \times [-1, 1]$  whose equivalence classes are the one point subsets  $(x, t)$  for  $t \neq \pm 1$  and the subsets  $X \times \{-1\}$  and  $X \times \{1\}$ , which are called the *poles* of the cone (sometimes the latter are called the south and north poles respectively and written  $P_-$  and  $P_+$ ).

(a) If  $f : X \rightarrow Y$  is a continuous mapping, explain why the mapping  $f \times \text{id}_{[-1,1]}$  from  $X \times [-1, 1]$  to  $Y \times [-1, 1]$  passes to a continuous map  $\Sigma(f) : \Sigma(X) \rightarrow \Sigma(Y)$  and this construction has the functorial properties  $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$  and  $\Sigma(\text{id}) = \text{id}$ .

(b) Prove that  $\Sigma(X) - \{P_-\}$  and  $\Sigma(X) - \{P_+\}$  are contractible, and in fact they are deformation retracts of  $\{P_+\}$  and  $\{P_-\}$  respectively. Also explain why  $\Sigma(X) - \{P_+, P_-\}$  is homotopy equivalent to  $X$ . [*Hint*: Why is it homeomorphic to  $X \times (-1, 1)$ ?]

(c) Prove that if  $X$  is arcwise connected then  $H_1(\Sigma(X)) = 0$ , while if  $q \geq 2$  then there is an isomorphism from  $H_q(\Sigma(X))$  to  $H_{q-1}(X)$  which is essentially given by the boundary map in a suitable Mayer-Vietoris exact sequence. [*Hint*: Use the decomposition

$$\Sigma(X) = \Sigma(X) - \{P_-\} \cup \Sigma(X) - \{P_+\}$$

and the conclusions of part (b) above. What is the intersection of the two given open subsets?]

**3.** Suppose that  $M_1$  and  $M_2$  are connected topological  $n$ -manifolds with  $p_i \in M_i$ , and let  $M_1 \vee M_2$  denote the quotient of the disjoint union  $M_1 \amalg M_2$  in which  $p_1$  and  $p_2$  are identified. — Informally, this space is a one point union of  $M_1$  and  $M_2$  which contains (homeomorphic copies of) the latter as closed subspaces, and the intersection of these subspaces is a single point.

(a) Explain why  $M_i \subset M_1 \vee M_2$  has an open neighborhood  $U_i$  such that  $M_i$  is a deformation retract of  $U_i$ . [*Hint*: Consider the subset  $M_1 \vee W_2$  where  $W_2$  is a neighborhood of  $p_2$  which is homeomorphic to an open  $n$ -disk, and do the same with the roles of 1 and 2 reversed.]

(b) Prove that if  $q > 0$  then  $H_q(M_1 \vee M_2)$  is isomorphic to  $H_q(M_1) \oplus H_q(M_2)$ .

**4.** (a) If  $X$  is a nonempty space prove that  $H_q(S^1 \times X)$  is isomorphic to  $H_q(X) \oplus H_{q-1}(X)$ . [*Hint*: Write  $S^1 = U \cup V$  where  $U - S^1 = \{1\}$  and  $V = S^1 - \{-1\}$ , consider the corresponding decomposition of  $S^1 \times X$ , and look at the associated Mayer-Vietoris exact sequence.]

(b) Using the same ideas and an induction argument, prove that  $H_q(S^n \times X)$  is isomorphic to  $H_q(X) \oplus H_{q-n}(X)$ . [*Hint*: Write  $S^n = U \cup V$  where  $U$  and  $V$  are complements of the “north and south poles”  $\pm \mathbf{e}_{n+1}$ , and observe that  $U \cap V \cong S^{n-1} \times \mathbb{R}$ .]

**5.** (a) Explain why  $H_0(X, A)$  is a free abelian group on the set of all arc components of  $X$  which do not contain any points of  $A$ .

(b) Compute the homology of  $(S^n, A)$  where  $n \geq 1$  and  $A$  is a finite set. There are two cases depending upon whether or not  $n = 1$ .

6. (a) Prove that the sphere  $S^m$  is not a retract of the sphere  $S^n$  if  $m \neq n$ .

(b) Suppose that  $A \subset X$  is a retract of  $X$ . Prove that  $H_q(X) \cong H_q(A) \oplus H_q(X, A)$  for all  $q \geq 0$ .

7. Let  $f : (D^n, S^{n-1}) \rightarrow (D^n, D^n - \{0\})$  be the inclusion map of pairs. Show that  $f$  defines homotopy equivalences from  $D^n$  to itself and from  $S^{n-1}$  to  $D^n - \{0\}$ , but  $f$  is not a homotopy equivalence of pairs. [Hint: Note that a homotopy equivalence of Hausdorff pairs  $(X, A) \rightarrow (Y, B)$  also defines a homotopy equivalence of pairs  $(X, \overline{A}) \rightarrow (Y, \overline{B})$ , where as usual  $\overline{C}$  denotes the closure of the subspace  $C$ .]

8. If  $d > 1$  is an integer, then the *generalized Möbius strip*  $M_d$  is defined to be the quotient of  $S^1 \times [0, 1]$  whose equivalence classes are one point sets corresponding to the points  $(z, t)$  where  $t < 1$  and subsets with  $d$  elements of the form  $\{(\alpha z, 1) \mid \alpha^d = 1\}$ . If  $d = 2$  this is just the usual Möbius strip.

(a) Let  $B \subset M_d$  denote the image of  $S^1 \times \{1\}$ . Prove that  $B$  is a deformation retract of  $M_d$ . [Hint: Show that the deformation retraction  $\rho$  of  $S^1 \times [0, 1]$  which pushes everything down vertically to  $S^1 \times \{1\}$  passes to a map  $M_d \rightarrow B$  and that the homotopy from the identity to the composite of  $\rho$  with the inclusion  $S^1 \times \{1\} \subset S^1 \times [0, 1]$  passes to a homotopy of the composite  $M_d \rightarrow B \subset M_d$ .

(b) Let  $q : S^1 \cong S^1 \times \{1\} \rightarrow B \subset M_d$  be induced by the quotient space projection from  $S^1 \times [0, 1]$  to  $M_d$ . Prove that there is a homeomorphism  $h$  from  $B$  to  $S^1$  such that the composite  $h \circ q$  is the map from  $S^1$  to itself which sends  $z$  to  $z^d$ . Also, explain why  $q$  is homotopic to the composite

$$S^1 \approx S^1 \times \{0\} \subset S^1 \times [0, 1] \rightarrow M_d$$

where the morphism on the right is just the quotient projection. [Hint for the first part: Show first that there is a continuous map  $h : B \rightarrow S^1$  such that  $h \circ q(z) = z^d$  by noting that if two points in  $S^1$  map to the same equivalence class in  $B$  then their images under the map  $z \rightarrow z^d$  are equal. Then verify that  $h$  is 1-1 onto and hence is a homeomorphism onto its image. — The second statement will follow easily from the fact that the slice inclusions  $i_0, i_1$  of  $S^1$  in  $S^1 \times [0, 1]$  are homotopic.]

(c) Let  $A_k \subset S^1$  be the minor arc whose endpoints are  $\beta^{k-1}$  and  $\beta^k$ , where  $\beta = \exp(2\pi i/3d)$ . Explain why  $A_k \times [0, 1]$  maps homeomorphically onto its image in  $M_d$ , and using the standard homeomorphisms  $A_k \times [0, 1] \cong [0, 1] \times [0, 1]$  construct a triangulation of  $M_d$  whose vertices correspond to the points  $(\beta^k, 0)$  and  $[\beta^k, 1]$ , where “[...]” denotes the associated equivalence class in  $M_d$ . Note that there are  $3d$  vertices of the first type and 3 of the second. [Hint: The 2-simplices should have vertices corresponding to triples of the forms  $\{(\beta^k, 0), (\beta^{k+1}, 0), [\beta^k, 1]\}$  and  $\{[\beta^k, 0], [\beta^k, 1], [\beta^{k+1}, 1]\}$ . Thus there are  $6d$  simplices of dimension 2, with  $3d + 3$  vertices and  $9d + 3$  edges. There is a drawing for this problem in the file `exercises04a.pdf`.]

(d) Let  $\mathbf{K}$  be the simplicial complex obtained in the preceding discussion, let  $\mathbf{L}$  be the subcomplex corresponding to  $S^1 \times \{0\}$ , and attach a cone  $\mathbf{Q}$  on  $\mathbf{L}$  whose simplices have the form  $uvv'$  and their boundaries, where  $vv'$  is an edge of  $\mathbf{L}$ . Let  $\mathbf{N}_d$  denote the union of  $\mathbf{K}$  and  $\mathbf{Q}$  along  $\mathbf{L}$ , and let  $P[d]$  denote the underlying space. Prove that  $H_q(P[d])$  is isomorphic to  $\mathbb{Z}$  if  $q = 0$ ,  $\mathbb{Z}_d$  if  $q = 1$ , and 0 otherwise. [Hint: Use a Mayer-Vietoris sequence for simplicial homology together with the previous observations that  $q_* : H_1(S^1) \rightarrow H_1(B)$  is multiplication by  $d$ , the inclusion map defines a homotopy equivalence from  $B$  to  $M_d$ , and  $q$  is homotopic to the composite  $S^1 \approx S^1 \times \{0\} \rightarrow S^1 \times [0, 1] \rightarrow M_d$ .]

**9.** For  $1 \leq q \leq n$  let  $G_n$  be a finitely generated abelian group. Prove that there is a connected polyhedron  $P$  such that  $H_q(P) = G_q$  for  $1 \leq q \leq n$ . [Hints: Use 8(d) and 3(b) to do the case where  $n = 1$ , and use 8(d), 3(b) and suspensions to get the conclusion in higher dimensions.]

**10.** (a) Using the Mayer-Vietoris sequence for the decomposition of  $T^2 - \{(-1, -1)\}$  into

$$(S^1 \times (S^1 - \{-1\})) \cup ((S^1 - \{-1\}) \times S^1)$$

prove that  $H_1$  of this arcwise connected space is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}$  in dimension 1 and is zero in all other positive dimensions. [Hint:  $S^1$  minus a point is contractible, as is the intersection of the open sets in the decomposition.]

(b) Using Exercise 4, compute the homology of  $T^2 = S^1 \times S^1$  and show that projection onto the two coordinates induces a surjection from  $H_1(T^2)$  to  $H_1(S^1)$ . [Hint: Explain why it suffices to consider projection onto the second coordinate and use some steps in the proof of Exercises 4.]

(c) Using the preceding observations, explain why the inclusion of  $T^2 - \{(-1, -1)\}$  in  $T^2$  induces an isomorphism in homology. [Hint: Let  $j_1, j_2$  denote inclusions of  $S^1$  as  $S^1 \times \{1\}$  and  $\{1\} \times S^1$  respectively, and let  $\pi_1, \pi_2$  denote the coordinate projections. Explain why  $\pi_{s*} \circ \pi_{t*}$  is the identity if  $s = t$  and trivial if  $s \neq t$ .]

(d) Let  $W$  be an open neighborhood of  $(-1, -1)$  in  $T^2$  such that  $W$  is homeomorphic to an open 2-disk with  $(-1, -1)$  corresponding to its center. By excision we know that  $H_*(W, W - \{(-1, -1)\}) \cong H_*(T^2, T^2 - \{(-1, -1)\})$ . Prove that the 2-dimensional generator of this group maps to zero in  $H_1(T^2 - \{(-1, -1)\})$  and hence lies in the image of  $H_2(T^2)$ . Using this show that the inclusion of  $W - \{(-1, -1)\} \cong S^1 \times (0, 1)$  in  $T^2 - \{(-1, -1)\}$  induces the zero map in 1-dimensional homology.

(e) The double torus or oriented surface of genus two has a decomposition of the form  $U_1 \cup U_2$  where  $U_i$  is homeomorphic to  $T^2 - \{(-1, -1)\}$  and the intersection is given by  $W_1 - \{(-1, -1)\} \subset U_1$  or equivalently  $W_2 - \{(-1, -1)\} \subset U_2$ . Compute the homology groups of the double torus using this decomposition and a Mayer-Vietoris sequence.

**11.** Suppose that  $U$  and  $V$  are open convex subsets of  $\mathbb{R}^n$  and  $U \cap V$  is nonempty. Prove that  $H_q(U \cup V) = 0$  if  $q \neq 0$  and  $H_0(U \cup V) \cong \mathbb{Z}$ , and give an example to show that  $U \cup V$  is not necessarily convex. [Hint:  $U \cap V$  is convex and if  $C$  is convex then  $C$  is contractible.]

**12.** ( $\star$ ) Suppose we are given a triple of spaces  $(X, A, B)$  such that  $A$  is a subspace of  $X$  and  $B$  is a subset of  $A$ . For each integer  $q$ , define the connecting homomorphism  $\partial_*$  from  $H_{q+1}(X, A)$  to  $H_q(A, B)$  to be the composite  $H_{q+1}(X, A) \rightarrow H_q(A) \rightarrow H_q(A, B)$ . Prove that the sequence

$$\cdots \rightarrow H_{q+1}(X, A) \rightarrow H_q(A, B) \rightarrow H_q(X, B) \rightarrow H_q(X, A) \rightarrow H_{q-1}(B, A) \cdots$$

is exact. [One reference for a proof is the book by Eilenberg and Steenrod.]