An alternate construction of free products

The following construction for free products may be a little simpler than the one in `fundgp-notes.pdf` (Theorem IX.2.4), and the proof given here is also less computational, relying instead on systematic use of universal mapping properties. For the sake of clarity we shall restate the result:

THEOREM IX.2.4. *If $\{G_\alpha \mid \alpha \in A\}$ is a nonempty indexed family of groups, then there exist data consisting of a group S and homomorphisms $i_\alpha : G_\alpha \to S$ which present S as a free product of the groups G_α.*

Proof. For each α we can find a free group F_α which is free on a set of generators X_α such that G_α is isomorphic to a quotient group F_α/N_α for some normal subgroup N_α; for each α let $q_\alpha : F_\alpha \to G_\alpha$ be the associated onto homomorphism of groups. Let $\mathbf{F}$ be a free group on the disjoint union of the sets X_α; then for each $\beta \in A$ the injection $e_\beta : X_\beta \to \amalg_\alpha X_\alpha$ extends uniquely to a group homomorphism $c_\alpha : F_\alpha \to \mathbf{F}$. Let $\mathbf{N} \subset \mathbf{F}$ be the normal subgroup which is normally generated by the union of the sets $c_\alpha[N_\alpha]$ where α runs through all elements of the indexing set A, let $S = \mathbf{F}/\mathbf{N}$, and let $q : \mathbf{F} \to \mathbf{F}/\mathbf{N}$ denote the quotient group projection. For each $\alpha \in A$ the composite homomorphism $d_\alpha = q \circ c_\alpha$ is trivial on N_α by construction, and therefore there is a unique group homomorphism $i_\alpha : G_\alpha \to S$ such that $i_\alpha \circ q_\alpha = d_\alpha = q \circ c_\alpha$.

CLAIM: The data data consisting of the group S and the homomorphisms $i_\alpha : G_\alpha \to S$ present S as a free product of the groups G_α.

The proof of this claim amounts to verifying that the data have the Universal Mapping Property described in `fundgp-notes.pdf`:

> *If H is a group and $\varphi_\alpha : G_\alpha \to H$ is an indexed family of homomorphisms, then* there is a unique group homomorphism $\Phi : S \to H$ *such that $\Phi \circ i_\alpha = \varphi_\alpha$ for all $\alpha \in A$.*

We shall construct the homomorphism in analogy with the construction of S. Let $X_\alpha \subset F_\alpha$ be the free generating set in that construction, and define $r_\alpha : X_\alpha \to H$ to be $\varphi_\alpha \circ q_\alpha | X_\alpha$. If $r : \amalg\, X_\beta \to H$ is defined by r_α on X_α for each α, then there is a unique extension of r to a group homomorphism $r' : \mathbf{F} \to H$ because $\mathbf{F}$ is free on $\amalg\, X_\beta$. By construction, for each α we have

$$r' \circ c_\alpha | X_\alpha \quad = \quad \varphi_\alpha \circ q_\alpha | X_\alpha$$

so by the uniqueness part of the Universal Mapping Property we have $r' \circ c_\alpha = \varphi_\alpha \circ q_\alpha$. Among other things, this implies that for each α the restriction $r' \circ c_\alpha | N_\alpha$ is trivial, and since $\mathbf{N}$ is normally generated by the subgroups $c_\alpha[N_\alpha]$ it follows that $r'|\mathbf{N}$ is also trivial.

The conclusion of the preceding sentence implies that there is a unique homomorphism $\Phi : S \to H$ such that $\Phi \circ q = r'$. We claim that $\Phi \circ i_\alpha = \varphi_\alpha$ for all α. Since the mappings q_α are onto, it is enough to show that for each α we have $\varphi_\alpha \circ q_\alpha = \Phi_\alpha \circ i_\alpha \circ q_\alpha$. By construction the right hand side is equal to

$$\Phi \circ q \circ c_\alpha \quad = \quad r' \circ c_\alpha \quad = \quad \varphi_\alpha \circ q_\alpha$$

and therefore the reasoning of the preceding sentence implies that $\Phi \circ i_\alpha = \varphi_\alpha$ for all α.

To conclude the proof, it suffices to show that if $\Phi' : S \to H$ satisfies $\Phi' \circ i_\alpha = \varphi_\alpha$ for all α then $\Phi = \Phi'$, and this will follow if S is generated by the union of the subsets $i_\alpha[G_\alpha]$. By construction we know that the free group $\mathbf{F}$ is generated by $\cup_\beta \; c_\beta[F_\beta]$, and since $q : \mathbf{F} \to S$ is onto it follows that S is generated by the set

$$q\left[\bigcup_\beta \; c_\beta[F_\beta]\right] \;=\; \bigcup_\beta \; q \circ c_\beta[F_\beta] \;=\; \bigcup_\beta \; i_\beta \circ q_\beta[F_\beta] \;=\; \bigcup_\beta \; i_\beta[G_\beta] \; .$$

Therefore S is indeed generated by the images of the mappings i_α.■