SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 7

Fall 2014

VII. Topological deformations and approximations

VII.0 : Categories and functors

Additional exercises

1. (a) Follow the hint and prove the contrapositives.

Not a monomorphism $\Rightarrow$ *not* $1-1$. If f is not a monomorphism, then there exist mappings $g, h : C \to A$ such that $g \neq h$ but $f \circ g = f \circ h$. Now $g \neq h$ means that there is some $x \in C$ such that $g(x) \neq h(x)$, and by hypothesis we have $f(g(x)) = f \circ g(x) = f \circ h(x) = f(h(x))$, so that f sends both $g(x)$ and $h(x)$ to the same element of B. But this means that f is not 1–1.■

Not an epimorphism $\Rightarrow$ *not onto.* If f is not an epimorphism, then there exist mappings $u, v : B \to D$ such that $u \neq v$ but $u \circ f = v \circ f$. The functional identity translates to the identity $u(f(a)) = v(f(a))$ for all $a \in A$, and therefore we have $u|f[A] = v|f[A]$. On the other hand, $u \neq v$ implies that $u(b) \neq v(b)$ for some $b \in B$, and by the previous sentence we know that b cannot belong to $f[A]$. Therefore $f[A]$ is a proper subset of B, which means that f is not onto.■

(b) Suppose that $f[A]$ is dense in B and $u \circ f = u \circ g$, where $u, v : B \to D$. Then u and v are equal on the dense subset $f[A]$. Since D is Hausdorff, the set E of all points b such that $u(b) = v(b)$ is closed. We know that E contains the dense subset $f[A]$, so we must have $E = B$.■

> *Note.* The *Wikipedia* article `http://en.wikipedia.org/wiki/Epimorphism` gives extensive information on the relationship between epimorphisms and surjective mappings for many standard examples of categories. Frequently, but not always, these notions are equivalent.

(c) Suppose we are given $f_1 : A \to B$ and $f_2 : B \to C$.

Assume both maps are monomorphisms. Let g and h be morphisms into A such that $(f_2 \circ f_1) \circ g = (f_2 \circ f_1) \circ h$. By associativity of composition and the monomorphism hypothesis on f_2, we have $f_1 \circ g = f_1 \circ h$; but now the monomorphism hypothesis on f_1 implies that $g = h$.■

Assume both maps are epimorphisms. Let u and v be morphisms from C such that $u \circ (f_2 \circ f_1) = v \circ (f_2 \circ f_1)$. By associativity of composition and the epimorphism hypothesis on f_1, we have $u \circ f_2 = v \circ f_2$; but now the epimorphism hypothesis on f_2 implies that $u = v$.■

(d) Suppose that f and g are morphisms $W \to X$ such that $r \circ f = f \circ g$, and let q be such that $q \circ r = \mathrm{id}_X$. Then we have

$$f \;=\; \mathrm{id}_X \circ f \;=\; q \circ r \circ f \;=\; q \circ r \circ g \;=\; \mathrm{id}_X \circ g \;=\; g$$

which means that r is a monomorphism.

(e) Suppose that u and v are morphisms $B \to D$ such that $u \circ p = v \circ p$. Then we have

$$u \;=\; u \circ \mathrm{id}_B \;=\; u \circ p \circ s \;=\; v \circ p \circ s \;=\; v \circ \mathrm{id}_B \;=\; v$$

which means that p is a epimorphism.

> *Note.* In the category of sets, the Axiom of Choice implies that every monomorphism is a retract and every epimorphism is a retraction, but for other categories this fails. For example, in the category of abelian groups the monomorphism $\mathbb{Z}_2 \to \mathbb{Z}_4$ is not a retract and the epimorphism $\mathbb{Z}_4 \to \mathbb{Z}_2$ is not a retraction. [PROOF: In the first case, if there was a homomorphism $q : \mathbb{Z}_4 \to \mathbb{Z}_2$ such that $q|\mathbb{Z}_2$ was the identity, then the fact that the image of $\mathbb{Z}_2$ is $2\mathbb{Z}_4$ implies that $q|2\mathbb{Z}_4$ is nonzero, and this cannot happen since $2\mathbb{Z}_2 = 0$. In the second case, if one could find a suitable homomorphism s then the image of s would be contained in $2\mathbb{Z}_4$ and once again this would yield the contradictory conclusion $p \circ s = 0$.]

2. We shall first dispose of the converse. Assume f is an isomorphism, and let $g : B \to A$ be its inverse. Then by associativity of composition we know that $\mathrm{Mor}(f, \cdots) \circ \mathrm{Mor}(g, \cdots)$ sends a morphism $h : A \to X$ to $h \circ g \circ f = h \circ \mathrm{id}_A = h$, so that $\mathrm{Mor}(f, \cdots) \circ \mathrm{Mor}(g, \cdots)$ is the identity on $\mathrm{Mor}(A, X)$. Similarly, by associativity of composition we know that $\mathrm{Mor}(g, \cdots) \circ \mathrm{Mor}(f, \cdots)$ sends a morphism $k : B \to X$ to $h \circ f \circ g = k \circ \mathrm{id}_B = k$, so that $\mathrm{Mor}(g, \cdots) \circ \mathrm{Mor}(f, \cdots)$ is the identity on $\mathrm{Mor}(B, X)$.

We shall now prove the exercise itself. Assume that $\mathrm{Mor}(f, \cdots)$ is an isomorphism from $\mathrm{Mor}(B, X)$ to $\mathrm{Mor}(A, X)$ for all X. If we let $X = A$ this means that there is a unique $p : B \to A$ such that $p \circ f = \mathrm{id}_A$. To prove the result we want, it suffices to show that $f \circ p = \mathrm{id}_B$. Since $\mathrm{Mor}(f, \cdots)$ is 1–1, it will suffice to prove that $(f \circ p) \circ f = \mathrm{id}_B \circ f = f$; but this follows immediately from $p \circ f = \mathrm{id}_A$.■

3. (*a*) For the empty set, the unique map is the one whose graph is the empty subset of $\emptyset \times A = \emptyset$ (this function is often called the *empty map*). For the one point set, the unique map is the one whose graph is the entire set $A \times \{\text{point}\}$ (in other words, the only possible constant map into the set).■

(*b*) Since a linear transformation always sends a zero vector to a zero vector, there is only one possibility for the map if the vector space consists only of the zero vector, so a zero space is an initial object. On the other hand, for a linear transformation into a zero space there is only possible value for the transformaton at a point; namely, the zero vector. Therefore a zero space is also a terminal object.■

(*c*) We shall first consider initial objects. Since there is an identity map from an initial object $\mathcal{O}$ to itself and by hypothesis there is only one self-map of $\mathcal{O}$, it follows that a map from $\mathcal{O}$ to itself must be the identity. Suppose now that $\mathcal{O}$ and $\mathcal{O}'$ are initial objects. Then there are unique maps $a : \mathcal{O} \to \mathcal{O}'$ and $b : \mathcal{O}' \to \mathcal{O}$. By the uniqueness of self-maps for initial objects, it follows that $b \circ a$ is the identity for $\mathcal{O}$ and $a \circ b$ is the identity for $\mathcal{O}'$. Therefore a and b are both isomorphisms.

We next consider terminal objects. Since there is an identity map from a terminal object $\mathcal{T}$ to itself and by hypothesis there is only one self-map of $\mathcal{T}$, it follows that a map from $\mathcal{T}$ to itself must be the identity. Suppose now that $\mathcal{T}$ and $\mathcal{T}'$ are terminal objects. Then there are unique maps $a : \mathcal{T} \to \mathcal{T}'$ and $b : \mathcal{T}' \to \mathcal{T}$. By the uniqueness of self-maps for terminal objects, it follows that $b \circ a$ is the identity for $\mathcal{T}$ and $a \circ b$ is the identity for $\mathcal{T}'$. Therefore a and b are both isomorphisms.■

4. If Z is a null object and A is an arbitrary object, then there are unique maps $A \to Z$ and $Z \to A$, so for each null object we obtain a unique composite morphism $A \to A$. We need to show that if W is any other null object, then the composites $A \to Z \to A$ and $A \to W \to A$ are equal. For the sake of definiteness, let $t_Z : A \to Z$ and $j_Z : Z \to A$ be the unique maps from and to the null object Z, and similarly for W. Now let $a : W \to Z$ and $b : Z \to W$ be the unique maps between the two null objects; then $a \circ b$ and $b \circ a$ must be identity maps by uniqueness, and consequently a and b are isomorphisms which are inverse to each other. Therefore we have

$$j_W \circ t_W \quad = \quad j_W \circ \mathrm{id}_W \circ t_W \quad = \quad j_W \circ b \circ a \circ t_W$$

and since $a \circ t_W = t_Z$ and $j_W \circ b = j_Z$ by uniqueness, it follows that $j_W \circ t_W = j_Z \circ t_Z$, and hence the composite does not depend upon a the choice of a specific null object.■

4. Suppose that $r : X \to Y$ is a retract and $s : Y \to X$ is a map such that $q \circ r = \mathrm{id}_X$. If F is a covariant functor defined on the category under consideration then we have

$$\mathrm{id}_{F(X)} \quad = \quad F(\mathrm{id}_X) \quad = \quad F(q \circ r) \quad = \quad F(q) \circ F(r)$$

and consequently $F(r)$ is also a retract. Similarly, if $p : A \to B$ is a retraction and $p \circ s = \mathrm{id}_B$, then we have

$$\mathrm{id}_{F(B)} \quad = \quad F(\mathrm{id}_B) \quad = \quad F(p \circ s) \quad = \quad F(p) \circ F(s)$$

and consequently $F(p)$ is also a retraction.

If G is a contravariant functor the conclusions are more complicated; namely, if r is a retract then $G(r)$ is a retraction, and if p is a retraction then $G(p)$ is a retract. The derivations are nearly the same as the preceding ones, but at the last step one must reverse the orders of composition.■

5. In fact, f is a retract, for there is a unique morphism $c : X \to E$ because E is terminal, and as noted before the morphism $c \circ f : E \to E$ must be the identity if E is terminal.■

6. This is basically a translation of fundamental statements about matrix multiplication into the language of category theory. A morphism $A : n \to m$ is merely an $m \times n$ matrix over the integers, the identity matrix is the identity morphism, and composition is matrix multiplication. The composition rules for identities and associativity are then merely restatements of the corresponding properties of matrix multiplication.■

7. Let $f : X \to Y$ be a morphism, and suppose that f has a quasi-inverse $g : Y \to X$; we claim there is some $h : Y \to X$ such that f is a quasi-inverse to h. The natural first candidate is $h = g$, but this does not lead anywhere so we need to find another choice for h. The correct choice is $g \circ f \circ g$, and the string of equations

$$(gfg)f(gfg) \quad = \quad (gfg)(fgf)g \quad = \quad (gfg)fg \quad = \quad g(fgf)g \quad = \quad gfg$$

shows that f is a quasi-inverse to $g \circ f \circ g$ (the composition operators were omitted in the display to make the equations easier to follow).■

> *Note.* Usually morphisms in a category do not have quasi-inverses, but in the category of sets the Axiom of Choice is essentially equivalent to the existence of quasi-inverses.

VII.1 : Homotopic mappings

Problems from Munkres, § 51*, p.* 330

2. On the unit interval $I = [0, 1]$ the identity map is homotopic to a constant map by convexity. Therefore we have the following:

(a) If $f : X \to I$ is continuous, then $f = \mathrm{id}_I \circ f$ is homotopic to $C_0 \circ f$, where C_0 is the constant function whose value everywhere is zero. Since $C_0 \circ f = C_0$, it follows that f is homotopic to the map which sends everything to zero.■

(b) If $f : I \to Y$ is continuous, then $f = f \circ \mathrm{id}_I$ is homotopic to $f \circ C_0$, which is the constant map with value $f(0)$ everywhere. Since Y is arcwise connected, one can use a curve joining $f(0)$ to an arbitrary point $y_0 \in Y$, and hence all constant maps are homotopic to each other. Combining these, we see that every continuous mapping $I \to Y$ is homotopic to the constant map with value y_0.■

3. (a) More generally, if $K \subset \mathbb{R}^n$, then the straight line homotopy $H(x,t) = (1-t)x + tx_0$ is a homotopy from the identity on K to the constant map whose value everywhere is x_0.■

(b) If $H : X \times [0,1] \to X$ is the homotopy from the identity to the constant map with value x_0, then $H|\{x\} \times [0,1]$ is a curve joining x to x_0, and therefore every point in X lies in the arc component of x_0.■

(c) This is essentially the same argument as in 2(a) with I replaced by an arbitrary contractible space.■

(d) This is essentially the same argument as in 2(b) with I replaced by an arbitrary contractible space.■

Problem from Munkres, § 52, p. 334

1. (a) If $A \subset \mathbb{R}^2$ is the union of the $x-$ and $y-$axes, then A is star convex with a_0 taken to be the origin because $x \in A$ and $0 < t < 1$ implies $t\,x \in A$. However, A is not convex because $(1,0)$ and $(0,1)$ are in A but $\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$ does not lie in A.■

(b) [*Not included because the concept of simple connectedness has not yet been introduced. However, another exercise has a stronger result; namely, a star convex set is contractible.*]

Additional exercises

1. The set of continuous maps $P \to X$ is in 1–1 correspondence with the points of X such that $f : P \to X$ corresponds to $f(p_0) \in X$. Two such continuous mappings are homotopic if and only if their values $f_0(p_0)$ and $f_1(p_0)$ can be joined by a continuous curve in X. Therefore one obtains a 1–1 correspondence between $[I, X]$ and the set of arc components of X by sending f to the arc component of $f(p_0)$.■

2. The idea is to follow the hint and prove that two of maps from X to Y are always homotopic. As noted in the hint, if Y has the indiscrete topology and W is an arbitrary topological space, then every map of sets from W to Y is continuous. In particular, if $V \subset W$ and $g : V \to Y$ is continuous, then there is an extension of g to a map of sets from W to Y, and this extension is automoatically continuous. In particular, this is true if $W = X \times [0,1]$ and $V = X \times \{0,1\}$, proving that if f_0 and f_1 are continuous mappings from X to the indiscrete space Y, then one can always construct a homotopy from f_0 to f_1.■

3. Since a continuous map takes connected sets to connected sets and the connected components of a discrete space are the one point subsets, it follows that every continuous map $X \to Y$ is a constant map and, in addition, every homotopy between two continuous maps is also constant. Therefore, if we take the 1–1 correspondence between points of Y and constant maps from X to Y, we obtain a map $Y \to [X, Y]$ which is both 1–1 and onto.■

4. (i) Star convexity implies that the image of the straight line homotopy $H(x,t) = (1-t)\,a_0 + t\,x$ is contained in A, so it follows that the identity on A is homotopic to the constant map whose value is a_0.$\blacksquare$

(ii) Follow the hint. First of all, if K and L are convex and $p \in K \cap L$, then $K \cup L$ is star convex with respect to p because $x \in K \cup L$ implies $x \in K$ or $x \in L$, and in either case the line segment defined by $(1-t)\,p + t\,x$ will be contained in $K \cup L$. In the example, K and L are convex subsets of $\mathbb{R}^2$, and one can check that $(7/4, 3/4)$ and $(3/4, 7/4)$ are in $A = K \cup L$. However, their midpoint is $(5/4, 5/4)$, and this point is not in A.$\blacksquare$

VII.2 : Some examples

Additional exercises

1. Let $x \in U$, and choose $\delta > 0$ such that $N_\delta(x)$ is contained in U. Since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, it follows that there is some point $y \in N_\delta(x) \cap \mathbb{Q}^n$. Now $N_\delta(x)$ is arcwise connected and hence is contained in the arc component of x, so we have shown that this arc component contains a point in $\mathbb{Q}^n$. It follows that we can define a 1–1 map from the arc components of U to $\mathbb{Q}^n$ by choosing some point in $\mathbb{Q}^n$ for each arc component. Since $\mathbb{Q}^n$ is countable, this means the set of arc components must also be countable.$\blacksquare$

2. Let $H : X \times [0,1] \to Y$ be the homotopy from f_0 to f_1, and consider the image B of $A \times [0,1]$ where A is an arcwise connected subset of X. Since $A \times [0,1]$ is arcwise connected, it follows that B is arcwise connected and hence is contained in an arc component of Y. The conclusion of the exercise now follows because both $f_0[A]$ and $f_1[A]$ are contained in B.$\blacksquare$

VII.3 : Homotopy classes of mappings

Problem from Munkres, § 58, pp. 366 – 367

6. The first thing to notice is that the arcwise connectedness of X implies that all constant maps into X are homotopic, and consequently if the identity on X is homotopic to a constant map, it is also homotopic to a constant map whose value lies in the subspace A. Now let $i : A \to X$ be a retract, with $r : X \to A$ such that $r \circ i = \mathrm{id}_A$, and let $H : X \times [0,1] \to X$ be a homotopy from the identity to a constant map whose value lies in A. Then the composite

$$h'(a,t) \quad = \quad r \circ H(a,t)$$

is a homotopy from the identity on A to a constant map.

Problems from Hatcher, pp. 18 – 20

4. Let $j : A \to X$ be the inclusion map, and let $g : X \to A$ be the map $g(x) = f_1(x)$, which exists because the image of f_1 is contained in A. By construction we then know that $j \circ g = f_1$ is homotopic to f_0, which is the identity on X. To prove that $g \circ j$ is homotopic to the identity on A, proceed as follows: Since each map f_t maps A into itself, it follows that f_t induces a homotopy

$H : A \times [0,1] \to A$ such that $H(a,t) = f_t(a)$ for all a and t. It then follows that H defines a homotopy from the identity on A to $f_1|A = j \circ g$.∎

10. We shall first prove the statement in the first sentence. If X is contractible, then the identity map on X is homotopic to a constant, so if $f : X \to Y$ is arbitrary then $f = f \circ \mathrm{id}_X$ is homotopic to $f \circ C_0$ for some constant map C_0; since the latter composite is also a constant map, this shows that f must be nullhomotopic. Conversely, the hypothesis $[X, Y] = \{\text{point}\}$ for all Y specializes to the case $Y = X$ and the statement that the homotopy class of the identity map in $[X, X]$ must be equal to the homotopy class of the constant map.∎

12. If $f : X \to Y$ is a homotopy equivalence and P is a one point set, then $f_* : [P, X] \to [P, Y]$ must be an isomorphism; since $[P, Z]$ is the set of arc components of Z by a previous exercise, it follows that f induces a 1–1 correspondence between the arc components of X and the arc components of Y.

The proof of the corresponding statement for connected components is also elementary but slightly different. We can express things formally as follows: If $g : A \to B$ is a continuous map, then for each connected component C of A we know that the connected set $g[A]$ must lie in a connected component of B, and therefore if **ConnComp**(E) denotes the set of connected components of a space E, then the continuous mapping g induces a map of sets

$$\mathbf{ConnComp}(g) : \mathbf{ConnComp}(A) \longrightarrow \mathbf{ConnComp}(B)$$

and it is an elementary exercise to verify that this defines a covariant functor on the category of topological spaces and continuous maps. Furthermore, since arcwise connected sets are connected, a variant on Additional Exercise VII.2.2 implies that if g_0 and g_1 are homotopic maps from A to B, then the induced maps of connected components satisfy the homotopy invariance property

$$\mathbf{ConnComp}(g_0) \quad = \quad \mathbf{ConnComp}(g_1) \ .$$

Combining these observations, we see that if f is a homotopy equivalence from X to Y then the map **ConnComp**(f) will be an isomorphism.

To prove the statement in the final sentence, observe that we have the following commutative diagram, in which we identify $[P, Z]$ with the arc components of a space Z and the horizontal maps send an arc component of a space Z to the connected component which contains it.

$$\begin{array}{ccc}
[P, X] & \xrightarrow{f_*} & [P, Y] \\
\downarrow \alpha_X & & \downarrow \alpha_Y \\
\mathbf{ConnComp}(X) & \xrightarrow{\mathbf{ConnComp}(f)} & \mathbf{ConnComp}(Y)
\end{array}$$

If f is a homotopy equivalence, then the results in the first two paragraphs imply that f_* and **ConnComp**(f) are isomorphisms. Therefore if either of the maps α_X or α_Y is an isomorphism, then so is the other.∎

Additional exercises

1. The key point to observe is that f and g are homotopy inverses to f^{-1} and g^{-1} respectively. Therefore it follows that $\mathrm{id}_X = f^{-1} \circ f \simeq f^{-1} \circ g$, yielding a relationship chain

$$g^{-1} \quad = \quad \mathrm{id}_X \circ g^{-1} \quad = \quad f^{-1} \circ g \circ g^{-1} \quad = \quad f{-1} \circ \mathrm{id}_X \quad = \quad f^{-1}$$

which shows that $f^{-1} \simeq g^{-1}$.■

2. (*i*) Follow the hints. For the first part, suppose that i is a retract. If $r : X \to A$ is such that $r \circ i = \mathrm{id}_A$, then by functoriality $r_* \circ i_* = (i = \mathrm{id}_A)_*$, and since the latter is the identity on $[A, Y]$ it follows that r_* is a retract, and consequently r_* is 1–1. For the second part, let $Y = A = \{0, 1\} \subset [0, 1] = X$ as in the hints. Since A is a discrete space, every self-map of A is continuous, and no two self-maps are homotopic (use a previous exercise), so the set $[A, A]$ contains exactly four elements. On the other hand, since X is contractible the set $[A, X]$ contains exactly one element by a previous exercise. Therefore the map $[A, A] \to [X, A]$ cannot be 1–1.■

(*ii*) In this case we have $i^* \circ r^* = (i = \mathrm{id}_A)^*$, and since the latter is the identity on $[Z, A]$ it follows that i^* is a retraction, and consequently i^* is onto. To show that i^* need not be onto, take $Z = A$ and X as in the first part of the exercise. Then $[A, X]$ consists of one point and $[A, A]$ consists of four points, and therefore $i^* : [A, X] \to [A, A]$ is not onto for this example.■

3. (*i*) In the setting of this exercise, a map $f : Z \to X \times Y$ is totally determined by $p \circ f$ and $q \circ f$; two maps are equal if and only if their coordinate projections are equal, and if we are given $Z \to X$ and $Z \to Y$ then this pair arises as the coordinate projections of a continuous map $Z \to X \times Y$. Similarly, if H is a homotopy from f to f', then $p \circ H$ and $q \circ H$ define homotopies of their respective coordinate projections, and if we have homotopies of the coordinate projections then they arise as the coordinate projections of some homotopy $Z \times [0, 1] \to Y$. These observations combine to prove that coordinate projections define an isomorphism θ_Z.

To prove the naturality property of θ_Z (*i.e.*, the commuative diagram), note that if $g : W \to Z$ is continuous, then the coordiante projections of $f \circ g$ are merely $p \circ f \circ g$ and $q \circ f \circ g$ respectively.■

(*ii*) Given homotopy classes $u, v, w \in [Y, X]$, choose representative continuous functions r, s, t respectively, and let C_1 denote the constant map $Y \to X$ whose value is always 1. The associative law on homotopy classes $u \cdot (v \cdot w) = (u \cdot v) \cdot W$ follows directly from the associativity identity

$$[r \cdot (s \cdot t)](y) \quad = \quad r(y)\, s(y)\, t(y) \quad = \quad [(r \cdot s) \cdot t](y)$$

which comes from the associativity of m and holds for all $y \in Y$. Similarly, $u \cdot [C_1] = u = [C_1] \cdot u$ follows from $r(y) \cdot 1 = r(y) = 1 \cdot r(y)$. Finally, an inverse to the homotopy class of u is given by the function $q(y) = r(y)^{-1}$, for we have $q \cdot r = C_1 = r \cdot q$ by the same reasoning as above.

If $h : Y \to Z$ is continuous, then h^* sends u, v, w to classes represented by $r \circ h$, $s \circ h$ and $t \circ h$. Therefore $h^*(u \cdot v)$ is represented by $(r \cdot s) \circ h$, which is equal to $(r \circ h) \cdot (s \circ h)$. Since the latter represents $h^*(u) \cdot h^*(v)$, it follows that h^* defines a group homomorphism with respect to the group structure on $[Y, X]$.

> *Note.* If $X = S^1$, then this group structure is abelian because S^1 is abelian, and the resulting abelian group — often called the *Bruschlinsky group* — is used in the file `polishcircle.pdf`.

4. (*a*) The straight line homotopy H_t from id_A to the constant map with value a_0 is a basepoint preserving homotopy. Therefore if $f : (A, a_0) \to (X, x_0)$ is a basepoint preserving map, then $f \circ H_t$ is a basepoint preserving homotopy from f to the basepoint preserving constant map.■

(*b*) Let H_t be as in part (*a*), and suppose we are given a basepoint preserving map $g : (X, x_0) \to (A, a_0)$. Then $H_t \circ g$ is a basepoint preserving homotopy from f to the basepoint preserving constant map.■

Note. In contrast to some previous exercises, there was no need to assume X was arcwise connected in (a) because a basepoint preserving map will always send the arcwise connected space A into the arc component of the basepoint in X.

5. (i) As suggested in the hint, let g be a homotopy inverse to f. Then the associated maps of homotopy classes $(f \circ g)_* = f_* \circ g_*$, $(g \circ f)_* = g_* \circ f_*$, $(f \circ g)^* = g^* \circ f^*$, and $(g \circ f)^* = f^* \circ g^*$ are all identity maps because $f \circ g$ and $g \circ f$ are homotopic to identity maps. It follows that g_* is an inverse to f_* and g^* is an inverse to f^*.

(ii) This was already done in Exercise 12 on page 19 of Hatcher, for which the solution was given above.■

6. The Cantor set X is given as the intersection of closed subsets X_n, where each X_n is a finite union of pairwise disjoint intervals, each of which has length 3^{-n}. If $u, v \in X$ lie in the same arc component, then for each n they must lie on one of these intervals and hence the distance between them is at most 3^{-n}, so that $|u - v| < 3^{-n}$ for all n. But this can happen only if $u = v$. Therefore the arc components of X are one point sets, and there are countably many of them. If X were homotopy equivalent to an open subset of some $\mathbb{R}^n$, then by an earlier exercise the open set and X would have to contain only countably many components. Consequently, there is no homotopy equivalence from X to an open subset of some $\mathbb{R}^n$.■

VII.4 : Homotopy types

Problem from Munkres, § 58*, pp.* 366 – 367

1. Note that "deformation retract" in Munkres means "strong deformation retract" in the sense of this course. We now proceed to the proof of the exercise.

Let $i : A \to B$ and $j : B \to X$ be the inclusions, and let $p : B \to A$ and $q : X \to B$ be the maps given by the deformation retract data. Then $(p \circ q) \circ (j \circ i) = p \circ (q \circ j) \circ i = p \circ \mathrm{id}_B \circ i = \mathrm{id}_A$, so it only remains to show that $(j \circ i) \circ (p \circ q)$ is homotopic to the identity relative to A. We know that $i \circ p$ is homotopic to the identity relative to A, and since $pq|A$ is the identity it follows that $(j \circ i) \circ (p \circ q)$ $(j \circ (i \circ p) \circ q)$ is homotopic to $(j \circ \mathrm{id}_A \circ q) = j \circ q$ relative to A. Since the right hand side is homotopic to the id_X relative to B and $A \subset B$, it follows that $(j \circ i) \circ (p \circ q)$ is homotopic to the identity relative to A. This means that A is a strong deformation retract of X.■

Problems from Hatcher, pp. 18 – 20

5. Let $H : X \times [0, 1] \to X$ be a homotopy from the identity to the constant map C_x, and let W be the open set $H^{-1}[W]$. This open subset contains $\{x\} \times [0, 1]$, so by Wallace's Theorem there is an open neighborhood V of x such that $V \times [0, 1] \subset W$. It follows that $H|V \times [0, 1]$ defines a homotopy into U from the inclusion $V \subset U$ to the constant map C_x on V.■

13. To simplify the notation, we shall denote the deformation retract data by $H : X \times [0, 1] \to X$ and $K : X \times [0, 1] \to X$ respectively. It will also be convenient to denote the boundary of $[0, 1] \times [0, 1]$ in $\mathbb{R}^2$ (the four edge segments) by Γ.

By the definition of deformation retract data, the homotopies H and K satisfy the following conditions:

(1) If $a \in A$, then $H(a,t) = K(a,t) = a$ for all $t \in [0,1]$.

(2) We have $H(x,0) = K(x,0) = x$ for all $x \in X$.

(3) We have $H(x,1) \in A$ and $K(x,1) \in A$ for all $x \in X$.

The goal of the exercise is to construct a homotopy from H to K; in other words, we want a map $L : X \times [0,1] \times [0,1] \to X$ with the following additional properties:

(4) We have $L(X,0,t) = H(x,t)$ and $L(X,1,t) = K(x,t)$ for all $x \in X$ and $t \in [0,1]$.

(5) If $a \in A$, then $L(a,s,t) = a$ for all $s,t \in [0,1]$.

(6) We have $L(x,s,1) \in A$ for all $x \in X$ and $s \in [0,1]$.

(7) We have $L(x,s,0) = x$ for all $x \in X$ and $s \in [0,1]$.

If L exists, then we can view the maps $L_s = L|X \times \{s\} \times [0,1]$ as a 1-parameter family of deformation retract data starting with H and ending with K.

The first step in constructing L is to define $D : X \times [0,1] \times [0,1] \to X$ by the formula

$$D(x,s,t) \quad = \quad H\,(\,K(x,s),\,t\,)\ .$$

The hypotheses on H and K imply that D has the following properties:

(8) We have $D(X,s,0) = K(x,s)$ and $D(X,0,t) = H(x,t)$ for all $x \in X$ and $s,t \in [0,1]$.

(9) If $a \in A$, then $L(a,t,s) = K(a,t,s) = a$ for all $s,t \in [0,1]$.

(A) We have $L(x,s,t) \in A$ and for all $x \in X$ if either $s = 1$ or $t = 1$.

(B) We have $D(x,0,0) = x$ for all $x \in X$.

Evidently the behavior of $D|X \times \Gamma$ does not fit the requirements for L that we have listed, but the properties of this restriction suggest that we can realize the requirements for L if we compose D with $\mathrm{id}_X \times \theta$, where $\theta : [0,1] \times [0,1] \to [0,1] \times [0,1]$ is a continuous map with the following behavior on Γ:

(C) The bottom edge $[0,1] \times \{0\}$ is collapsed to $(0,0)$.

(D) The left edge $\{0\} \times [0,1]$ is mapped to itself by the identity.

(E) The top edge $[0,1] \times \{1\}$ maps to the union of the top and right edges $[0,1] \times \{1\} \cup \{1\} \times [0,1]$ such that $[0, \frac{1}{2}] \times \{1\}$ maps to the top edge such that $(t,1)$ is sent to $(2t,1)$ and $[\frac{1}{2},1] \times \{1\}$ maps to the right edge such that $(t,1)$ is sent to $(1, 2-2t)$.

(F) The right edge $\{1\} \times [0,1]$ maps to the bottom edge $[0,1] \times \{0\}$ such that $(1,t)$ is sent to $(t,0)$.

The drawing in `math205Aexercises7a.pdf` illustrates the behavior we want on Γ, and a suitable function θ is constructed in that document. One can then use the definitions to verify that the function $L(x;s,t) = D(x;\theta(s,t))$ has all the desired properties.■

Additional exercises

1. In words, the subset A consists of the bottom edge and the side edges of the rectangle X. As usual, we shall follow the approach in the hint; there is a drawing for this exercise in `math205Aexercises7a.pdf`.

The retraction $r : [-1,1] \times [0,1] \to [-1,1] \times \{0\} \;\cup\; \{-1,1\} \times [0,1]$ is defined by a radial projection with center $(\mathbf{0}, 2) \in [-1,1] \times \mathbb{R}$. As indicated by the drawing, the formula for r depends upon whether $2|x| + t \geq 2$ or $2|x| + t \leq 2$. Specifically, if $2|x| + t \geq 2$ then

$$r(x,\,t) \quad = \quad \frac{1}{|x|}\left(\,x,\,2|x| + t - 2\right)$$

while if $2|x| + t \leq 2$ then we have

$$r(x,\,t) \quad = \quad \frac{1}{2}\left(\,(2-t)x,\,0\,\right)$$

and these are consistent when $2|x|+t = 2$ then both formulas yield the value $|x|^{-1}(x,\,0)$. Elementary but slightly tedious calculation then implies that $r(x,\,t)$ always lies in $[-1,1] \times [0,1]$, and likewise that r is the identity on $[-1,1] \times \{0\} \;\cup\; \{-1,1\} \times [0,1]$. The homotopy from inclusion$\circ r$ to the identity is then the straight line homotopy

$$H(x,\,t;\,s) \quad = \quad (1-s)\cdot r(x,\,t) \;+\; s \cdot (x,\,t)$$

and this completes the proof of the exercise.■

2. Follow the hint: Let $X' \subset X \times [0,1]$ be the subset $X \times \{0\} \cup A \times [0,1]$, and let $A' = A \times \{1\}$.

If $h : X \to X'$ is the composite $X \cong X \times \{0\} \subset X'$, let $\rho : X' \to X$ maps $X \times \{0\}$ to X by projection onto the first coordinate and maps $A \times [0,1]$ to A similarly. Then Exercise 4 implies that $X \times \{0\}$ is a deformation retract of X' and hence h is a homotopy equivalence. Take the mapping h_0 to be the slice embedding $A \cong A \times \{1\}$. A homotopy from $h|A$ to $j \circ h_0$ is given by $K(a,t) = (a,t)$. Finally, we need to find an open neighborhood U of A' in X' such that A' is a strong deformation retract of U, and this can be done by taking $U = A \times (0,1]$. Since $A' = A \times \{1\}$ it follows that A' is a strong deformation retract of U, so the only thing remaining is to prove that U is open in X'. We can see this most easily by viewing everything as a subspace of $X \times [0,1]$; in particular, since $V = X \times (0,1]$ is open in $X \times [0,1]$ and $X' \cap V = A \times (0,1] = U$, it follows that U is open in X'.■

3. For each α we have mappings $g_\alpha : Y_\alpha \to X_\alpha$ such that $g_\alpha \circ f_\alpha$ is homotopic to the identity on X_α and $f_\alpha \circ g_\alpha$ is homotopic to the identity on Y_α; denote these homotopies by H_α and K_α respectively.

We claim that $\prod_\alpha \; g_\alpha$ is a homotopy inverse to $\prod_\alpha \; f_\alpha$, and this requires the construction of homotopies

$$\Phi : \left(\prod_{\alpha \in A} X_\alpha\right) \times [0,1] \;\longrightarrow\; \prod_{\alpha \in A} X_\alpha\;, \qquad \Psi : \left(\prod_{\alpha \in A} Y_\alpha\right) \times [0,1] \;\longrightarrow\; \prod_{\alpha \in A} Y_\alpha$$

such that the composites

$$\prod_{\alpha \in A} g_\alpha \circ f_\alpha \quad = \quad \left(\prod_{\alpha \in A} g_\alpha\right) \;\circ\; \left(\prod_{\alpha \in A} f_\alpha\right)$$

$$\prod_{\alpha \in A} f_\alpha \circ g_\alpha \quad = \quad \left(\prod_{\alpha \in A} f_\alpha\right) \;\circ\; \left(\prod_{\alpha \in A} g_\alpha\right)$$

are respectively homoptopic to the identity maps on $\prod_{\alpha \in A} X_\alpha$ and $\prod_{\alpha \in A} Y_\alpha$.

As usual, it suffices to define the coordinate projections of Φ and Ψ for each $\gamma \in A$, and we do so by the formulas

$$\pi_\gamma \circ \Phi = H_\gamma \circ \left(\pi_\gamma \times \mathrm{if}_{[0,1]}\right) , \qquad \pi_\gamma \circ \Psi = K_\gamma \circ \left(\pi_\gamma \times \mathrm{if}_{[0,1]}\right) .$$

It follows immediately that these maps define homotopies from

$$\left(\prod_{\alpha \in A} g_\alpha\right) \circ \left(\prod_{\alpha \in A} f_\alpha\right) \quad \text{and} \quad \left(\prod_{\alpha \in A} f_\alpha\right) \circ \left(\prod_{\alpha \in A} g_\alpha\right)$$

to the identity mappings.∎

4. (*i*) Let $r : F \to B$ and $h : F \times [0,1] \to F$ be the deformation retraction data which exist because B is a strong deformation retract of F. Extend r to $r' : X \to A$ by letting $r'|A$ be the identity, and extend h to $H : X \times [0,1] \to X$ by letting $H|A \times [0,1]$ be projection onto the first factor. If these maps are continuous, then they for deformation retract data for $A \subset X$. But the continuity of these extensions follows because $F \cap A = B$, and the restriction of r to B is the identity, and the restrictions of h to $B \times [0,1]$ is projection onto the first factor.∎

(*ii*) For $i = 1, 2$ let r_i and $H^{(i)}$ be deformation retract data for $C \subset F_i$. Then the restrictions of these data to X and $X \times [0,1]$ are the identity and projection onto the first factor respectively, so we can assemble r_1 and r_2 into a continuous mapping $r : X \to C$, and likewise we can assemble $H^{(1)}$ and $H^{(2)}$ into a continuous mapping $H : X \times [0,1] \to X$. The properties of the deformation retract data for $C \subset F_i$ imply that r and H are deformation retract data for $C \subset X$.∎

Drawing for Hatcher, Exercise 13, page 19

This drawing is meant to motivate and clarify a crucial step in our solution for the cited exercise. We need to find a continuous mapping from the solid square $[0, 1] \times [0, 1]$ to itself with the following behavior on the edges:

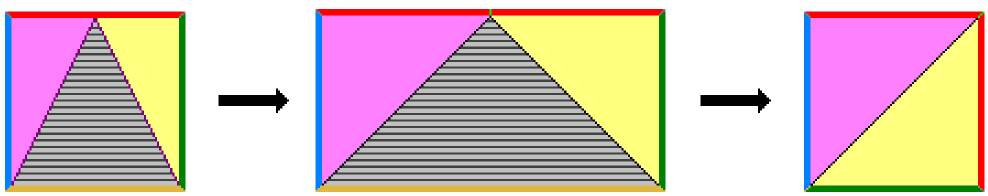
(1) The bottom edge (in ***gold***) is collapsed to the corner point $(0, 0)$.

(2) The left edge (in ***blue***) is mapped to itself.

(3) The top edge (in ***red***) maps to the union of the top and right edges.

(4) The right edge (in ***green***) maps to the bottom edge.

The drawing and discussion below suggest one way of constructing such a function.



It is convenient to break the mapping into a twofold composite, where the first step is to stretch the square $[0, 1] \times [0, 1]$ to the rectangle $[0, 2] \times [0, 1]$ in the obvious fashion. The second step is to collapse the rectangle back into the square, such that the (pink) triangle with vertices $(0, 0)$, $(0, 1)$, and $(1, 1)$ is mapped to itself by the identity, the (yellow) triangle with vertices $(1, 1)$, $(2, 1)$, and $(2, 0)$ is mapped to the congruent triangle with vertices $(1, 1)$, $(1, 0)$, and $(0, 0)$ such that $(1, 1)$ corresponds to itself, $(2, 1)$ corresponds to $(1, 0)$, and $(2, 0)$ corresponds to $(0, 0)$; also, the (striped gray) triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$ is collapsed along horizontal lines onto the closed segment joining the first two vertices. Explicit formulas defining this mapping are given as follows; to simplify the notation we shall refer to the triangles by their colors (we have already defined them formally):

(1) On the pink triangle, which consists of the points (u, v) in the rectangle such that $u \leq v$, the map sends (u, v) to itself.

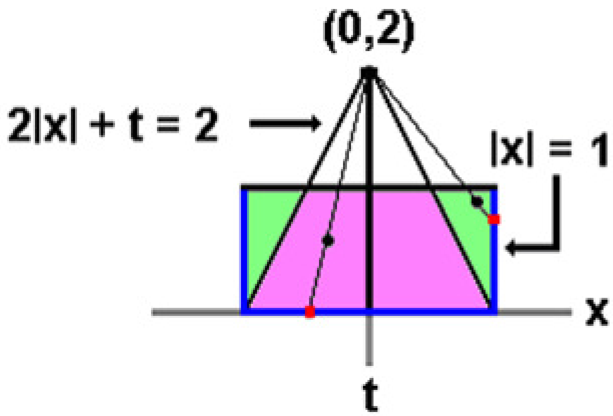
(2) On the striped gray triangle, which consists of the points (u, v) in the rectangle such that $v \leq u \leq 2 - v$, the map sends (u, v) to (v, v).

(3) On the yellow triangle, which consists of the points (u, v) in the rectangle such that $2 - v \leq u$, the map sends (u, v) to $(v, 2 - u)$.

The first two formulas are easy to derive, and the third can be done using linear algebra in various ways. For example, this can be done by finding the unique affine transformation of the coordinate plane which takes the points $(1, 1)$, $(2, 1)$, and $(2, 0)$ to the points $(1, 1)$, $(1, 0)$, and $(0, 0)$ respectively, or one can derive the formula by observing that the map on the yellow triangle is a clockwise rotation through 90 degrees which is centered at $(1, 1)$. Either way, the formula for the mapping turns out to be the one which is given in the third case. One can check directly that these definitions yield the same values for points on more than one triangle, so the formulas yield a well defined function.

Drawing to accompany Additional Exercise VII.4.1

Here is a picture of the retraction described in the hint:



In this illustration, the retraction from $[-1,1] \times [0,1]$ to $[-1,1] \times \{0\} \cup \{-1,1\} \times [0,1]$ (the blue line segments) sends the points marked in black into the points marked in red on the corresponding lines. The explicit definition of the retraction has two cases, depending upon whether or not the original point lie in the pink colored region or the green colored region(s).

A similar argument shows that $\mathbf{D}^n \times [0,1] \cup \mathbf{S}^{n-1} \times [0,1]$ is a retract of $\mathbf{D}^n \times [0,1]$ for all positive integers n. For example, one can obtain the case $n = 2$ from the $1-$dimensional case by taking solids and surfaces of revolution about the $t-$axis, and likewise in higher dimensions one can view the drawing as a planar cross section of the general construction.

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 8

Fall 2014

VIII. The fundamental group

VIII.1 : Definitions and basic properties

Problems from Munkres, § 52, *pp.* 334 – 335

2. We shall start by translating the conclusion into the notation in the course notes. Given a curve θ, Munkres' notation $\widehat{\theta}$ refers to the curve we call $-\theta$; for a curve defined on $[0,1]$ we have $-\theta(t) = \theta(1-t)$. So in our terminology the identity to be verified is

$$-(\alpha \ + \ \beta) \ \ = \ \ (-\beta) \ + \ (-\alpha) \ .$$

Recall that $\{\alpha+\beta\}(t) = \alpha(2t)$ if $t \leq \frac{1}{2}$ and $\{\alpha+\beta\}(t) = \beta(2t-1)$ if $t \leq \frac{1}{2}$. Therefore $-\{\alpha+\beta\}(t) = \{\alpha+\beta\}(1-t)$ is given by

(*a*) $\alpha(2(1-t)) = \alpha(2-2t)$ if $(1-t) \leq \frac{1}{2}$ or equivalently if $t \geq \frac{1}{2}$,

(*b*) $\beta(2(1-t)-1) = \beta(1-2t)$ if $(1-t) \geq \frac{1}{2}$ or equivalently if $t \leq \frac{1}{2}$.

Similarly, we see that $\{(-\beta) \ + \ (-\alpha)\}(t)$ is given by

(*c*) $-\beta(2t) = \beta(1-2t)$ if $t \leq \frac{1}{2}$,

(*d*) $-\alpha(2t-1) = \alpha\big(\,1-(2t-1)\,\big) = \alpha(2-2t)$ if $t \geq \frac{1}{2}$.

Therefore we have shown that for each $t \in [0,1]$ the values of $-(\alpha+\beta)$ and $(-\beta) \ + \ (-\alpha)$ at t are the same, and hence the two curves are equal.■

4. Let $i : A \to X$ denote the inclusion map. Since the fundamental group construction defines a covariant functor, we have $r_* \circ i_* = \mathrm{id}_{\pi_1(A)}$. Therefore if $u \in \pi_1(A, a_0)$ we have $u = r_*(i_*(u))$ and therefore u lies in the image of r_*, which means that r_* is surjective.■

7. (*a*) In order to prove functional identities, one needs to show that the values of both sides of the equation at every point s in the domain are the same. We apply this to verify the associativity, neutral element and inverse identities in $\Omega(G,1)$:

Associativity. For all s we have

$$\{(f \otimes g) \otimes h\}(s) \ = \ \big(f(s) \cdot g(s)\big) \cdot h(s) \ = \ f(s) \cdot \big(g(s) \cdot h(s)\big) \ = \ \{f \otimes (g \otimes h)\}(s) \ .$$

Neutral element. If $C_1(t) = 1$ for all t, then for all s we have

$$\{f \otimes C_1\}(s) \ = \ f(s) \cdot 1 \ = \ f(s) \ , \qquad \{C_1 \otimes f\}(s) \ = \ 1 \cdot f(s) \ = \ f(s) \ .$$

Inverses. If $g(t) = f(t)^{-1}$ for all t, then for all s we have

$$\{f \otimes g\}(s) \;=\; f(s) \cdot g(s) \;=\; 1 \;=\; C_1(s)\ , \qquad \{g \otimes f\}(s) \;=\; g(s) \cdot f(s) \;=\; 1 \;=\; C_1(s)\ .$$

(b) The crucial point to verify is that if f_0 and g_0 are endpoint preserving homotopic to f_1 and g_1 respectively, then $f_0 \otimes g_0$ is endpoint preserving homotopic to $f_1 \otimes g_1$. If we know this, then we can define a binary operation on $\pi_1(G, 1)$ by noting that there is a well defined binary operation on the latter with $[f] \otimes [g] = [f \otimes g]$. The associativity, neutral element and inverse identities will then follow from the corresponding identities derived in (a).

To prove the statement in the preceding paragraph, note that if H and K are endpoint preserving homotopies from f_0 and g_0 to f_1 and g_1 respectively, then $H \otimes K$ is endpoint preserving homotopy from $f_0 \otimes g_0$ to $f_1 \otimes g_1$.■

(c) Follow the hint. Direct computation yields the identity

$$f \;+\; g \quad=\quad (f \;+\; C_1) \otimes (C_1 \;+\; g)$$

from which we find that $[f] \cdot [g] = [f + C_1] \otimes [C_1 + g] \;=\; [f] \otimes [g]$.■

(d) For each value of s either $\{f + C_1\}(s)$ or $\{C_1 + g\}(s)$ is equal to 1, so these two curves commute with respect to the "$\otimes$" operation. Once again applying the reasoning in (c), we find that $[f] \otimes [g] = [g] \otimes [f]$ for all $[f]$ and $[g]$. The main conclusion of (c) now implies that $[f[\cdot[g] = [f] \otimes [g] = [g] \otimes [f] = [g] \cdot [f]$.■

Problems from Hatcher, pp. $38 - 40$

10. This is very similar to parts of the preceding exercise. Let $i_X : (X, x_0) \to (X \times Y, (x_0, y_0))$ be the slice inclusion sending x to (x, y_0), and let $i_Y : (Y, y_0) \to (X \times Y, (x_0, y_0))$ be the slice inclusion sending y to (x_0, y). The goal is to construct a homotopy from $(i_X \circ f) + (i_Y \circ g)$ to $(i_Y \circ g) + (i_X \circ f)$. As is always the case with mappings into products, it is enough to construct the homotopies for the coordinate projections of these maps onto (X, x_0) and (Y, y_0). In other words, we only need to construct homotopies

$$p_X \circ \big((i_X \circ f) + (i_Y \circ g) \big) \ \text{ to } \ p_X \circ \big((i_Y \circ g) + (i_X \circ f) \big)$$

$$p_Y \circ \big((i_X \circ f) + (i_Y \circ g) \big) \ \text{ to } \ p_Y \circ \big((i_Y \circ g) + (i_X \circ f) \big)$$

(where p_X and p_Y are coordinate projections), and we shall explain how these may be found using homotopies we have already constructed (for our puposes, this is "explicit" enough).

Since $p_X \circ i_X = \mathrm{id}_X$, $p_Y \circ i_Y = \mathrm{id}_Y$, $p_X \circ i_Y = \text{constant}(x_0)$ and $p_Y \circ i_X = \text{constant}(y_0)$, we can translate the display to conclude that we only need to construct homotopies from $f + \text{constant}(x_0)$ to $\text{constant}(x_0) + f$ and from $g + \text{constant}(y_0)$ to $\text{constant}(y_0) + g$. This can be done by splicing together the standard homotopies from $h + \text{constant}$ to h and from h to $\text{constant} + h$ for $h = f$ or g.■

13. As stated, the problem has an almost trivial solution which require NO HYPOTHESES on the map of fundamental groups or the arcwise connectedness of A. Here is the solution: Given any curve $\gamma : [0, 1] \to X$ with endpoints in A, the homotopy $h_s(t) = \gamma(st)$ defines a homotopy from the constant curve with value $\gamma(0) \in A$ to the original curve γ. Presumably the author intended the following, which we shall prove below: ... iff every path in X with endpoints in A is *endpoint preserving* homotopic to a curve in A.

($\Longrightarrow$) Suppose that $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is onto, and let $a_0, a_1 in A$. Since A is arcwise connected, there are continuous curves α_0 and α_1 joining x_0 to a_0 and a_1 respectively.

We claim that every curve $\gamma : [0, 1] \to X$ such that $\gamma(i) = a_i$ for $i = 0, 1$ is endpoint preserving homotopic to a curve of the form $(-\alpha_0 + \beta) + \alpha_1$, where β is a basepoint preserving closed curve in X. In fact, if we set β equal to $(\alpha_0 + \gamma) + (-\alpha_1)$, then we have the identities

$$[\gamma] \quad = \quad [C(a_0) + \gamma + C(a_1)] \quad = \quad [(-\alpha_0) + \alpha_0 + \gamma + (-\alpha_1) + \alpha_1] \quad = \quad [(-\alpha_0 + \beta) + \alpha_1]$$

when we adopt the convention in the course notes that endpoint preserving homotopy classes of iterated concatenations do not depend upon where parentheses are inserted. The hypothesis in this part of the problem is that β is endpoint preserving homotopic to a curve θ whose image lies in A, and therefore $[\gamma] = [(-\alpha_0) + \beta + \alpha_1]$ is equal to

$$(-\alpha_0 + \theta) + \alpha_1$$

, where the image of the representative curve $(-\alpha_0 + \theta) + \alpha_1$ is contained in A.$\blacksquare$

($\Longrightarrow$) This follows immediately, for if γ is a closed curve in X which is endpoint (hence basepoint) preserving homotopic to a curve θ whose image is in A, then $[\gamma] = i_*([\theta])$.$\blacksquare$

Additional exercises

1. Parts of this exercise are similar to parts of Exercise 52.7 in Munkres, for which a solution is given above.

(i) The first statement to prove is that the map sending (α, β) to $\alpha + \beta$ is continuous. We can do this by showing that $\mathbf{d}(\alpha' + \beta', \alpha + \beta) < \varepsilon$ if $\mathbf{d}(\alpha', \alpha) < \varepsilon$ and $\mathbf{d}(\beta', \beta) < \varepsilon$. In fact, the maximum distance between the points $\{\alpha' + \beta'\}(s)$ and $\{\alpha + \beta\}(s)$ is the greater of (a) the maximum distance between the points $\{\alpha'\}(s)$ and $\{\alpha\}(s)$, (b) the maximum distance between the points $\{\beta'\}(s)$ and $\{\beta\}(s)$.

We now turn to the homotopy assertions, and we shall use the notation of Proposition VIII.1.3. Let $H_R : [0, 1] \times [0, 1] \to [0, 1]$ be the straight line homotopy from the identity to h_R, and let $H_L : [0, 1] \times [0, 1] \to [0, 1]$ be the straight line homotopy from the identity to h_L. Define corresponding maps $K_R, K_L : \Omega(X, x) \times [0, 1] \to \Omega(X, x)$ by $\{K_R(\gamma, t)\}(s) = \gamma \circ H_R(s, t)$ and $\{K_L(\gamma, t)\}(s) = \gamma \circ H_L(s, t)$. The definitions then imply that $K_L(\gamma, 0) = \gamma \circ h_L = C_x + \gamma$ and $K_L(\gamma, 1) = \gamma$, and similarly we know that $K_R(\gamma, 0) = \gamma \circ h_R = \gamma + C_x$ and $K_R(\gamma, 1) = \gamma$. If γ' is another element of $\Omega(X, x_0)$, then by construction the distance between $K_R(\gamma, t)$ and $K_R(\gamma', t)$ is equal to the distance between γ and γ', and likewise for $K_R(\gamma, t)$ and $K_R(\gamma', t)$. These imply that K_L and K_R are continuous and hence yield homotopies of the types described in the statement of the exercise.

(ii) Follow the hint and imitate the reasoning in Munkres, Exercise 52.7. The results of (i) show that $(Y, e) = (\Omega(X, x), C_x)$ is an H-space as defined following the statement of this exercise. We shall prove more generally that $\pi_1((Y, e)$ is abelian using the approach for the exercise in Munkres. Let $m : Y \times Y \to Y$ denote the continuous binary operation, and define a binary operation "$\otimes$" on $\pi_1(Y, e)$ sending $([f], [g])$ to the class of the composite $f \otimes g(s) \;=\; m\big(f(s), g(s)\big)$. We shall often denote the right hand side by $f(s) \cdot g(s)$ for the sake of simplicity, but when we do so we have to remember that this construction does not necessarily satisfy associativity or neutral element identities; all we know is that f is basepoint preservingly homotopic to both $f \cdot C_e$ and $C_e \cdot f$. The latter suffice to yield two weaker identities

$$[f \;+\; g] \quad = \quad [f \;+\; C_1] \otimes [C_1 \;+\; g]\ , \qquad [f \;+\; g] \quad = \quad [C_1 \;+\; g] \otimes [f \;+\; C_1]$$

and since the right hand sides are equal to $[f] \otimes [g]$ and $[g] \otimes [f]$ respectively, it follows that the "$\otimes$" operation agrees with the usual group operation on the fundamental group, and both operations are abelian.■

(*iii*) If $\varepsilon > 0$, then there is some $\delta > 0$ such that $\mathbf{d}(x, x') < \delta$ implies $\mathbf{d}(\,f(x),\ f(x')\,) < \varepsilon$. It follows that if α and β are curves such that $\mathbf{d}(\alpha, \beta) < \delta$, then $\mathbf{d}(f \circ \alpha, f \circ \beta) < \varepsilon$ (because the distance between two curves is the maximum distance between their values at points of the domain).

(*iv*) The second part follows from the first because continuous mappings from one compact metric space to another are always uniformly continuous, so everything reduces to proving the first assertion in the conclusion. For $n \geq 2$, define $\pi_n(X) = \pi_1(\Omega^{n-1}(X, x), \text{basepoint})$ as in the statement of the exercise. By (*ii*) we know this is an abelian group. By (*iii*) and induction we know that a basepoint preserving uniformly continuous mapping f induces a map with the same properties, say $\Omega^{n-1}(f)$, from $\Omega^{n-1}(X, x)$ to $\Omega^{n-1}(Y, y)$, and this construction is functorial because it is given by composition of functions. Define $f_* : \pi_n(X, x) \to \pi_n(Y, y)$ to be the homomorphism of fundamental groups induced by $\Omega^{n-1}(f)$.■

Note. Hatcher and most other books covering homotopy theory define π_n differently, but eventually one almost always finds a proof that their construction(s) is/are equivalent to the one given here.

2. We claim that basepoint preserving maps from $(S^0, 1)$ to (X, x) are the same as maps from $\{-1\}$ to X and basepoint preserving homotopies are the same as homotopies of such mappings $\{-1\} \to X$. The first part is true because the basepoint of S^0 must go to the basepoint of X, but there are no constraints on where the second point can go. To see the statement on homotopies, note that the restriction of a homotopy to $\{1\} \times [0, 1]$ must be constant but the restriction to $\{-1\} \times [0, 1]$ can be an arbitrary continuous mapping from $[0, 1]$ into X.■

VIII.2 : An important special case

Problems from Munkres, § 58, pp. 366 – 367

2. (*a*) Infinite cyclic.

(*c*) Infinite cyclic.

(*d*) Infinite cyclic.

(*f*) Infinite cyclic.

(*g*) Infinite cyclic.

(*h*) Trivial.

(*i*) Infinite cyclic.

(*j*) Infinite cyclic.

9. (*a*) *This was not included because our definition does not involve any choices of an initial point on the circle.*

(*b*) As in the statement of the exercise, let $\omega(t) = \exp 2\pi\, i\, t$, and let $t_0 \in \mathbb{R}$ be such that $p(t_0) = h \circ \omega(0)$, where $p : \mathbb{R} \to S^1$ is the usual map $p(t) = \exp 2\pi\, i\, t$. Let α be the unique path lifting of $h \circ \omega$ starting at t_0. Then *deg* (h) is the unique integer $d(h)$ such that $\alpha(1) = t_0 + d(h)$.

Now let $H : S^1 \times [0,1] \to S^1$ be a homotopy from h to k, and let $L : [0,1] \times [0,1] \to \mathbb{R}$ be the unique lifting of H such that $L(0,0) = t_0$ and $p \circ L(s,t) = H(p(s),t)$. By the uniqueness of path lifings we know that $\alpha(s) = L(s,0)$ and the curve $L(s,1)$ is a lifting β of k. Furthermore, since $L(1,t)$ and $L(0,t)$ are both liftings of the curve $H|\{1\} \times [0,1]$. it follows that there is some integer Δ such that $L(1,t) = L(0,t) + \Delta$ for all t. In particular, we have $L(1,1) = L(0,1) + \Delta$ and $L(1,0) = L(0,0) + \Delta$. Now $L(s,0) = \alpha(s)$ by the uniqueness of path liftings, and therefore $\Delta = \mathit{deg}\,(h)$. On the other hand, $L(s,1)$ is a lifting of $k(s) = H(s,1)$, and therefore it also follows that $\Delta = \mathit{deg}\,(k)$. In words, we have shown that (freely) homotopic maps from S^1 to itself have the same degree.∎

(c) This one is probably easier to prove using the functoriality properties of the fundamental group. The proof of the main theorem in this section immediately yields the following result:

CLAIM. *If f is a basepoint preserving continuous mappings from S^1 to itself (with say $f(1) = 1$), then the self-homomorphism f_* on $\pi_1(S^1, 1) \cong \mathbb{Z}$ is multiplication by $\mathit{deg}\,(f)$.*

Proof of the claim. Since the class $[\omega]$ generates $\pi_1(S^1, 1)$, it is enough to prove that $f_*([\omega]) = \mathit{deg}\,(f) \cdot [\omega]$, and we can do this using the Path Lifting Property because the image $\partial(f)$ of $f_*([\omega]) = [f \circ \omega]$ in $\mathbb{Z}$ is given by taking a lifting θ of $f \circ \omega$ starting at $0 \in \mathbb{R}$ and setting $\partial(f)$ equal to $\theta(1)$. This coincides with the definition of degree for the mapping f.∎

Before proceeding, we note that the claim yields the conclusion of the exercise if h and k are basepoint preserving, for in that case we have

$$\deg(h \circ k) \cdot [\omega]) \quad = \quad (h \circ k)_*([\omega]) \quad = \quad (h_* \circ k_*)([\omega]) \quad = \quad h_*\big(\deg(k) \cdot [\omega]\big) \quad =$$

$$\deg(k) \cdot h_*([\omega]) \quad = \quad \deg(k) \cdot \deg(h) \cdot [\omega]$$

and since $[\omega] \in \pi_1(S_1, 1) \cong \mathbb{Z}$ has infinite order it follows that $\deg(h \circ k) = \deg(h) \cdot \deg(k)$.

In order to apply the preceding discussion to h and k, we have to replace them with (freely) homotopic mappings which ARE basepoint preserving. The easiest way to do this is to choose a and b such that $p(a) = h(1)$ and $p(b) = k(1)$ and define homotopies by $H(z,t) = h(z) \cdot p(ta)^{-1}$ and $K(z,t) = k(z) \cdot p(tb)^{-1}$; it then follows that $H_0 = h$ and H_1 is basepoint preserving, and similarly $K_0 = k$ and K_1 is basepoint preserving. Applying (ii) and recalling that $H_1 \circ K_1$ is homotopic to $h \circ k$, we find that

$$\deg(h \circ k) \quad = \quad \deg(H_1 \circ K_1) \quad = \quad \deg(H_1) \cdot \deg(H_1) \quad = \quad \deg(h) \cdot \deg(k)$$

which is what we wanted to prove.∎

Note. The tools developed in 205B yield a simpler proof which generalizes to a notion of degree for continuous self maps of S^n when $n \geq 2$.

(d) The unique lifting α of a constant map f is a constant map, so $\alpha(1) = \alpha(0)$ implies that the degree is zero. — For the identity, we know that ω is a unique lift, and here the degree is $\omega(1) - \omega(0) = 1$. — For z^n where n is an integer, we know that a unique lifting is given by $\omega_n(t) = nt$, and in this case $\omega_n(1) - \omega_n(0) = n$ is the degree; this applies to each of the final two cases in the exercise.∎

(e) Suppose that h and k have the same degree, and let H_1 and K_1 be basepoint preserving maps which are freely homotopic to h and k respectively. Since $h \simeq H_1$ and $k \simeq K_1$, it suffices to show that if h and k have the same degree (so by (i) the same is true for H_1 and K_1), then H_1 and K_1 are homotopic. But the latter follows directly from the proof of the main result, for if the degrees are equal then $\partial(H_1) = \partial(K_1)$, and hence the unique liftings of H_1 and K_1 starting at 0 are endpoint preserving homotopic. We can then compose this homotopy with p to obtain a homotopy from H_1 to K_1, and since homotopy of maps is an equivalence relation it will follow that $h \simeq k$.■

Problems from Hatcher, pp. $38 - 40$

3. ($\Longrightarrow$) Suppose that X is arcwise connected and $\pi_1(X, x_0)$ is abelian. If α and β are two continuous curves joining x_0 to $x_1 \in X$, we want to prove that $\alpha^* = \beta^*$. — The desired conclusion is equivalent to proving that $(\alpha + (-\beta))^*$ induces the identity on $\pi_1(X, x_0)$ for all α and β, and if we make the substitution $\gamma = \alpha + (-\beta)$ this translates into proving that for each closed curve γ which starts and ends at x_0 the automorphism γ^* — which we have shown is given by $\gamma^*(u) = [\gamma]^{-1}\, u\, [\gamma]$ — is the identity. This identity holds if and only if $\pi_1(X, x_0)$ is abelian, so $\alpha^* = \beta^*$ in this case.

($\Longrightarrow$) Suppose now that we have $\alpha^* = \beta^*$ for all paths α and β from x_0 to x_1. By the same reasoning as in the first part, it follows that $\gamma^*(u) = [\gamma]^{-1}\, u\, [\gamma]$ is the identity for all closed curves γ and $u \in \pi_1(X, x_0)$. In other words, we have $u = v^{-1}uv$ for all u and v in the fundamental group. But this is true if and only if $\pi_1(X, x_0)$ is abelian.■

16. If $A \subset X$ and a retraction $r : X \to A$ exists, then the induced map in fundamental groups $\pi_1(A) \to \pi_1(X)$ is 1–1 and the induced map r_* in fundamental groups is onto, so it suffices to prove that either $\pi_1(A) \to \pi_1(X)$ is not 1–1 or $\pi_1(A)$ is not isomorphic to a homomorphic image of $\pi_1(X)$.

(a) If $X = \mathbb{R}^3$ and $A \cong S^1$ then $\pi_1(X)$ is trivial but $\pi_1(A) \cong \mathbb{Z}$. Since a nontrivial group cannot be isomorphic to a homeomorphic image of a trivial group, there cannot be a retraction.■

(b) This is a little informal, in part because the inclusion $A \to X$ is only described by means of an unannotated drawing, but it can be justified if one describes the inclusion $A \subset X$ explicitly; this turns out to be possible, but at this stage it would take several paragraphs to explain everything in full detail. In this case both $\pi_1(A)$ and $\pi_1(X)$ are infinite cyclic, and if $B \subset X = S^1 \times D^2$ is equal to $S^1 \times \{\mathbf{0}\}$, then the inclusion of $A \cong S^1$ in X is homotopic to the map $S^1 \to B \cong S^1$ given by something like $p \circ f(t)$, where $f : [0, 1] \to \mathbb{R}$ is the closed curve given by $f(t) = (1 + h)\sin t$ for some small $h > 0$. By convexity f is homotopic to a constant curve in $\mathbb{R}$, so $p \circ f$ is homotopic to a constant curve in B, and hence the mapping $A \to B$ is null homotopic. But this means that the composite $\pi_1(A) \to \pi_1(B) \cong \pi_1(X)$ is trivial. Since $\pi_1(A)$ and $\pi_1(X)$ are infinite cyclic, this means that the map $\pi_1(A) \to \pi_1(X)$ is not 1–1, and we have noted that in such cases a retraction cannot exist.■

17. The assignment modifies the question to ask for infinitely many homotopy classes of retractions for the slice inclusion $j_1 : S^1 \to S^1 \times S^1$ sending z to $(z, 1)$. At the end of the exercise we shall explain how one can extract a solution to the exercise as it is stated in Hatcher.

Given an integer n, let $r_n : S^1 \times S^1 \to S^1$ be the map sending (z, w) to $z\, w^n$. It follows that for each n we have $r_n \circ j_1 = \mathrm{id}$. To prove that r_n and r_m are not homotopic if $m \neq n$, let $j_2 : S^1 \to S^1 \times S^1$ be the other slice inclusion $j_2(z) = (1, z)$. Then the degree of $r_n \circ j_2$ is equal to n, and therefore the homotopy classes of the mappings r_n and r_m are distinct if $m \neq n$.■

Solution for the problem as stated in Hatcher. In this case $S^1 \vee S^1$ is identified with the subspace

$$S^1 \times \{1\} \ \cup\ \{1\} \times S^1 \quad \subset \quad S^1 \times S^1$$

and the slice inclusions j_1, j_2 factor through maps i_1, i_2 from S^1 to $S^1 \vee S^1$. Therefore if $r'_n = r_n | S^1 \vee S^1$ we have $r'_n \circ i_1 = \mathrm{id}$ and $\deg\,(r'_n \circ i_2) = n$.■

Additional exercises

0. This was done already as a step in the solution to Munkres, Exercise 58.9 (*q.v.*).■

1. Follow the hint. Given continuous mappings $f, g : S^1 \to S^1$, with product $h(z)$, let $\alpha, \beta : [0,1] \to \mathbb{R}$ be liftings and let γ be their algebraic sum (if we write this symbolically, it will conflict with the use of "+" for concatenation); as suggested by the hint, the curve γ is a lifting of h. By definition the degree of h is equal to $\gamma(1) - \gamma(0)$, and the latter is equal to

$$\alpha(1) + \beta(1) - \alpha(0) - \beta(0) \quad = \quad \big(\alpha(1) - \alpha(0)\big) + \big(\beta(1) - \beta(0)\big) \quad = \quad \deg(f) + \deg(g)$$

which is what we wanted to prove.■

2. (*i*) It is probably best to start by describing $\Pi(X)$ more explicitly. Its objects are the points of X, and a morphism from x_0 to x_1 is an endpoint preserving homotopy class of curves from x_0 to x_1. The identity morphism for x is just the homotopy class of the constant curve at x, and if α and β are curves joining x_0 to x_1 and x_1 to x_2 respectively, then the formal composite $[\beta] \circ [\alpha]$ is merely $[\alpha + \beta]$ (note the order reversal!). The morphisms from x to itself then correspond to elements of $\pi_1(X, x)$ (with reversed multiplication!), and $[\alpha]^{-1} = [-\alpha]$ by the results of Sections VIII1 and VIII.2.

Functoriality can now be seen fairly easily. On objects, each point $x \in X$ goes to $f(x) \in Y$, and the homotopy class of a curve α joining x_0 to x_1 goes to the homotopy class $f_*([\alpha])$ of $f \circ \alpha$. The results of this and the previous section imply that this does not depend on which α we choose to represent an endpoint preserving homotopy class. Since $f \circ \alpha$ is constant if α is constant, it follows that the identity morphism for x is sent to the identity morphism for $f(x)$, and the chain of identities

$$[f \circ \beta]\,[f \circ \alpha] \quad = \quad [f \circ (\alpha + \beta)] \quad = \quad f_*([\alpha + \beta]) \quad = \quad f_*([\beta][\alpha])$$

shows that the construction preserves composition of morphisms.■

3. Before starting, we recall the construction of $J_k : \pi_1(T^k, \mathbf{1}) \to \mathbb{Z}^k$: Let $p_k : \mathbb{R}^k \to T^k$ be the Cartesian k^{th} power map $p \times \cdots \times p$ (k factors) from $\mathbb{R}^k$ to T^k. Let $\mathbf{1} \in T^k$ be the point whose coordinates are all equal to the unit element of S^1.

If γ is a basepoint preserving closed curve in T^k which starts and ends at 1, the we can apply the Path Lifting and Covering Homotopy Properties to each coordinatefunction of γ; this yields a unique lifting of γ to a curve $\Gamma : [0,1] \to \mathbb{R}^k$ such that $\Gamma(0) = \mathbf{0}$, and $\Gamma(1) \in \mathbb{Z}^n$ because $p \circ \Gamma(1) = \mathbf{1}$. As in the proof of the main result, $\Gamma(1)$ only depends upon the basepoint preserving homotopy class of γ, and hence the map $\gamma \to \Gamma(1)$ defines a map $J_k : \pi_1(T^k, \mathbf{1}) \to \mathbb{Z}^k$; the argument proving the main result of this section also shows that J_k is a group isomorphism. The inverse map is given by sending $(d_1, \cdots, d_k)$ to the homotopy class of the curve $(p(d_1 t), \cdots, p(d_k t))$, where $p : \mathbb{R} \to S^1$ is the usual map.

(*i*) If $(a, b) \in \mathbb{Z}^2$, then $m_* \circ J_2^{-1}$ sends (a, b) to the curve $p(at) \cdot p(bt) = p((a + b)t)$, and J_1^{-1} takes the latter curve to $a + b \in \mathbb{Z}^2$.■

(*ii*) Suppose that f is given as in the exercise. For each j between 1 and k let $s_j : S^1 \to T^k$ denote the slice inclusion onto the j^{th} slice (all coordinates are 1 except possibly the j^{th} one). Then the images of the homomorphisms s_{j*} generate $pi_1(T^k, \mathbf{1})$, so it is enough to compute the degree of each map $f \circ s_j$. The latter is the map sending z to z^{c_j}, and therefore the degree is equal to c_j. This immediately yields the conclusion of part (*ii*).■

4. (*i*) Since we are mapping into a product, it is enough to prove this for projection onto the factors, and we have already done this in the second part of the preceding exercise.■

(*ii*) If Φ_A is a homeomorphism, or even homotopic to a homeomorphism, then the induced homomorphism Φ_{A*} of $\pi_1(T^k, \mathbf{1}) \cong \mathbb{Z}^k$, which corresponds to left multiplication by A on $\mathbb{Z}^k$, must be an automorphism, and therefore $\det A$ must be equal to ± 1. Conversely, if $\det A = \pm 1$ and $B = A^{-1}$, then by Cramer's Rule B has integral entries and hence we can construct Φ_B. The definitions of the maps imply that $\Phi_{AB} = \Phi_A \circ \Phi_B$ and similarly if B and A are reversed; furthermore, if A is the identity matrix I then it follows that Φ_I is the identity mapping. These observations combine to imply that Φ_B is inverse to Φ_A.■

(*iii*) The first part follows because an arbitrary self-homomorphism (*endomorphism*) of the group $\pi_1(T^k, \mathbf{1}) \cong \mathbb{Z}^k$ is given by some $k \times k$ matrix A with integral entries, and we have shown that Φ_{A*} corresponds to left multiplication by A. The second part follows because if this matrix A is 1–1 and onto then Φ_A is a basepoint preserving homeomorphism from T^k to itself.■

VIII.3 : Covering spaces

Problems from Munkres, § 53, *p.* 341

For these exercises, we do NOT assume the Default Hypothesis.

1. If p is given as in the exercise, then for each $x \in X$ the whole space X is an evenly covered open neighborhood of x in X.■

2. We know that $p * -1[U]$ is homeomorphic to $U \times F$ for some discrete space F. Since U is connected, this means that the sheets $U \times \{y\}$ are the connected components of $U \times F$, and under the homeomorphism they correspond to the connected components of $p * -1[U]$.■

4. Let $z \in Z$, and let U be an open neighborhood of $z \in Z$ which is evenly covered; specifically, let $r^{-1}[U]$ be the union of the pairwise disjoint open subsets $U_1, \cdots, U_n$ such that r maps each U_j homeomorphically onto U, and let $y_j \in U_j$ be the unique point such that $r(y_j) = z$. Since q is a covering space projection, for each j there is an open neighborhood V_j of y_j in Y which is evenly covered. Intersecting V_j with U_j if necessary, we might as well assume that $V_j \subset U$ (an open subset of an openly covered open set is also evenly covered!). Let $W = \cap_j \, r[V_j]$. Then by construction $z = r(y_j)$ (for each j) lies in W, and this set is open in Z because r is an open mapping. Furthermore, W is evenly covered by the union of the pairwise disjoint subsets W_j. Now let $W_j = r^{-1}[W] \cap V_j$, so that $y_j \in W_j$ and W_j is also evenly covered. Therefore, for all j the inverse image $q^{-1}[W_j]$ is homeomorphic to $W_j \times F_j$, where F_j is discrete, such that the restriction of q to $q^{-1}[W_j]$ corresponds to projection onto W_j.

For each j let $r_j : W_j \to W$ denote the homeomorphism determined by r. Since $p = r \circ q$, it follows that

$$p^{-1}[W] \quad = \quad q^{-1}\left[r^{-1}[W]\right] \quad = \quad q^{-1}\left[\bigcup_j W_j\right] \quad \cong$$

$$\coprod_j W_j \times F_j \quad \cong \quad W \times \left(\coprod_j F_j\right)$$

where we use the homeomorphisms $W_j \cong W$ at the last step. This implies that the open neighborhood W of z is evenly covered with respect to p, and the latter means that p is also a covering space projection.■

6. (*a*) Let $p : X \to Y$ be a covering space projection. This part of the exercise involves proving that if Y has a stated topological property, then so does X. There are several distinct properties, and we shall verify the assertions about them separately. Note that the priorities on the parts of this exercise vary depending upon the specific property.

Hausdorff. Let $x_1 \neq x_2$ in X. There are two cases depending upon whether or not $p(x_1) = p(x_2)$. If $p(x_1) \neq p(x_2)$, then since Y is Hausdorff there are disjoint open neighborhoods U_1 and U_2 of $p(x_1)$ and $p(x_2)$ in Y, and their inverse images $p^{-1}[U_1]$ and $p^{-1}[U_2]$ are disjoint open neighborhoods of x_1 and x_2 in X. On the other hand, if $p(x_1) = p(x_2)$, let W be an evenly covered open neighborhood of this point. Then $p^{-1}[W]$ is an open set homeomorphic to a disjoint union of copies of W. Since $x_1 \neq x_2$, one of these copies contains x_1 and another contains x_2, and these two copies of W in X are disjoint open neighborhoods of the two points.■

Regular. Let $x \in X$. It suffices to prove the regularity condition for a set of open neighborhoods $\mathcal{V}$ of x such that every open neighborhood of X contains a subneighborhood in $\mathcal{V}$, for if $x \in W$ open and $x \in V \subset W$ with $V \in \mathcal{V}$, then $x \in U \subset \overline{U} \subset V$ implies the same inclusions with W replacing V.

In view of the preceding discussion, let $x \in V$, where V is an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Since Y is regular, there is an open neighborhood U of $p(x)$ such that $p(x) \in U \subset \overline{U} \subset W$. Let $U_1 = p^{-1}[U] \cap V$ be the sheet over U which is contained in V. If we can prove that the closure $\overline{U_1}$ of U_1 in X is contained in V, then the defining condition for regularity will hold at the point x, and since x was chosen arbitrarily this will imply that X is regular.

It will suffice to prove that $p^{-1}\left[\overline{U}\right] \cap V$, which contains U_1, is closed in X because we would then have $\overline{U_1} \subset p^{-1}\left[\overline{U}\right] \cap V \subset V$. By construction and continuity we know that $p^{-1}\left[\overline{U}\right]$ is closed in X and it is contained in the evenly covered open subset $p^{-1}[W]$. If V^* denotes the union of all sheets in $p^{-1}[W]$ except X, then V^* is open and hence $X - V^*$ is closed in X; with this notation we can rewrite the inclusion in the preceding sentence in the form $p^{-1}\left[\overline{U}\right] \subset V \cup V^*$, and it follows that

$$p^{-1}\left[\overline{U}\right] \cap V \quad = \quad p^{-1}\left[\overline{U}\right] \cap (X - V^*)$$

and since the right hand side is an intersection of two closed subsets of X, it follows that the left hand side is also a closed subset of X, which is what we needed to complete the proof.

Completely regular. By the preceding discussion we know that X is regular, and as in the preceding discussion it suffices to prove that if V is an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Let U be an open subneighborhood such that $x \in U \subset \overline{U} \subset V$, so that $U' = p[U]$ and $V' = p[V]$ are open neighborhoods of $p(x)$ such that $p(x) \in U' \subset \overline{U'} \subset V'$. Since a subspace of a completely regular space is regular, there is a continuous function $g : V' \to [0,1]$ such that $g(p(x)) = 0$ and $g = 1$ on $V' - U'$; the composite $g \circ p$ satisfies similar properties: The value of $g \circ p$ at x is zero and $g \circ p = 1$ on $V - U$. If we define a new function $f : Xto[0,1]$ such that $f|\overline{U} = g \circ p|\overline{U}$ and $f|X - U = 0$, then these functions agree on the overlapping closed subset $\overline{U} - U$, and therefore f defines a continuous function on X which is 1 at x and 0 off U.■

Locally compact Hausdorff. We already know that X is Hausdorff, so it is only necessary to show that every $x \in X$ has some open neighborhood whose closure is compact. As before, let V be an open neighborhood of x such that $p[V] = W$ is evenly covered and V is one of the sheets. Since Y is locally compact, it follows that there is some open neighborhood U of $p(x)$ such that $\overline{U} \subset V$ and $\overline{U}$ is compact. Let $U_1 = p^{-1}[U] \cap V$ be the sheet over U which is contained in V, and let $F = p^{-1}\left[\overline{U}\right] \cap V$. Since $p|V$ is a homeomorphism onto an open subset, it follows that F is homeomorphic to $\overline{U}$ and hence is compact. Therefore we have found an open neighborhood U_1 of x and a compact subset $F \subset X$ such that $x \in U_1 \subset F \subset V$. Since X is compact we know that F is closed in X, and therefore it follows that $\overline{U_1}$ is also compact. As before, we started with an arbitrary $x \in X$, so the argument implies that X is locally compact near every point and hence is a locally compact (Hausdorff) space.■

(b) We shall first prove this when Y is Hausdorff (in which case X is also Hausdorff by part (a).

Let $p : X \to Y$ be a covering space projection such that Y is compact and there are only finitely many sheets at each point of Y. We shall prove that X is a union of finitely many compact subsets. Let $y \in Y$, and let V_y be an open neighborhood of y which is evenly covered. Since a locally compact Hausdorff space is regular, there is some subneighborhood W_y of V_y such that $\overline{W_y} \subset V_y$ (and $\overline{W_y}$ is compact). The family $\mathcal{W}$ of all sets W_y is an open covering of Y, so there is a finite subcovering $\{W_{y_1}, \cdots, W_{y_k}\}$. Since the covering has finitely many sheets, each of the sets $p^{-1}\left[\overline{W_{y_j}}\right]$ is also compact, and since these form a finite closed covering of X it follows that X is also compact.

Here is a proof when Y is not necessarily Hausdorff.

Let $\mathcal{U}$ be an open covering of X. For each $y \in Y$, let V_y be an evenly covered open subset, and let $V_{y,1}, \cdots V_{y,k(y)}$ denote the sheets over V_y (we know there are only finitely many). If $x_j \in V_{y,j}$ is the unique point such that $p(x_j) = y$, then there is an open subneighborhood $\Omega_{y,j}$ of x_j which is contained in some open set U_α in the open covering $\mathcal{U}$. Since p is an open mapping, the set $W_y = \cap_j \, p\left[\Omega_{y,j}\right]$ is an evenly covered open neighborhood of y which is contained in V_y. If $W_{y,j} = p^{-1}[W_y] \cap V_{y,j}$, then p maps this set homeomorphically onto the open neighborhood W_y.

Since Y is compact, the open covering $\mathcal{W}$ of Y by the open subsets W_y has a finite subcovering consisting of sets W_z, where z lies in some finite subset $Z \subset Y$. The sheets of the inverse images $p^{-1}[W_z]$ then form an open covering of X such that each set in this open covering is contained in some U_α which belongs to $\mathcal{U}$. Therefore if for each $W_{z,j}$ we choose some $U_{\alpha(z,j)}$ such that $W_{z,j} \subset U_{\alpha(z,j)}$, then the open sets $U_{\alpha(z,j)}$ form a finite subcovering of $\mathcal{U}$.■

Problem from Hatcher, pp. $79 - 82$

2. Assume that we are given covering space projections $p_1 : E_1 \to B_1$ and $p_2 : E_2 \to B_2$; we need to prove that $p_1 \times p_2$ is also a covering space projection.

Let $(x_1, x_2) \in B_1 \times B_2$. Then the hypotheses imply that for $i = 1, 2$ there is an evenly covered open neighborhood U_i of X_i; *i.e.*, there are discrete spaces A_i and B_i together with homeomorphisms $h_i : p_i^{-1}[U_i] \to U_i \times A_i$ such that $\mathrm{proj}(U_i) \circ h_i$ is the restriction of p_i to $p_i^{-1}[U_i]$. We claim that $U_1 \times U_2$ is an evenly covered neighborhood of (x_1, x_2) in $B_1 \times B_2$. This is true because the homeomorphism

$$H : (p_1 \times p_2)^{-1}[U_1 \times U_2] \;=\; p_1^{-1}[U_1] \times p_2^{-1}[U_2] \;\longrightarrow\; U_1 \times A_1 \times U_2 \times A_2 \;\cong\; (U_1 \times U_2) \times (A_1 \times A_2)$$

(where the last map switches the second and third factors) is such that $\mathrm{proj}(U_1 \times U_2) \circ H$ is the restriction of $p_1 \times p_2$ to $(p_1 \times p_2)^{-1}[U_1 \times U_2]$.■

1. (i) Let U be an open subset of B, and consider the following commutative diagram, which is derived from the exercise by considering various restriction mappings:

$$\begin{array}{ccc} p^{-1}[U] & \xrightarrow{\Phi_U} & f^{-1}\left[\varphi[U]\right] \\ \big\downarrow p & & \big\downarrow f_{\varphi[U]} \\ U & \xrightarrow{\varphi_U} & \varphi[U] \end{array}$$

Let $y \in Y$, set x equal to $\varphi^{-1}(y)$, and let U be an evenly covered open neighborhood of x, so that there is a homeomorphism $h : p^{-1}[U] \to U \times A$ satisfying the defining identity. Then the composite

$$h' \quad = \quad (\varphi_U \times \mathrm{id}) \circ h \circ \Phi_U^1$$

is a homeomorphism from $f^{-1}\left[\varphi[U]\right]$ to $\varphi[U] \times A$ such that $f_{\varphi[U]}$ is h' followed by coordinate projection onto $\varphi[U]$. Therefore $\varphi[U]$ is an evenly covered open neighborhood of y.∎

(ii) Let $x \in X$, and choose U_α in $\mathcal{U}$ such that $x \in U_\alpha$. Since q_α is a covering space projection, we know that there is some open neighborhood $V \subset U_\alpha$ of x such that V is evenly covered by q_α, and since q_α has the same values as p at all points (but a different domain and codomain) it follows that V is also evenly covered by p.∎

2. Given $x \in X$ we can find open neighborhoods U_1 and U_2 of x which are evenly covered by p_1 and p_2, and by local connectedness there is a connected subneighborhood $U \subset U_1 \cap U_2$; it follows that U is evenly covered with respect to both p_1 and p_2. If $\mathcal{U} = \{U_\gamma \mid \gamma \in \Gamma\}$ is an open covering of E_2 by open subsets which are connected and evenly covered by p_1 and p_2, then by part (ii) of Exercise 1 it will suffice to show that the restricted maps

$$p_1^{-1}[U_\gamma] = p^{-1}\left[p_2^{-1}[U_\gamma]\right] \;\longrightarrow\; p_2^{-1}[U_\gamma]$$

determined by p are all covering space projections.

If U is connected and evenly covered by p_1 and p_2, let Let $h_1 : p_1^{-1}[U] \to U \times A$ and $h_2 : p_2^{-1}[U] \to U \times B$ be homeomorphisms (where A and B are discrete spaces) such that $\mathrm{proj}(U) \circ h_i$ is the restriction of p_i, and let $q : U \times A \to U \times B$ be the map unique continuous mapping from $p_1^{-1}[U] = p^{-1}\left[p_2^{-1}[U]\right]$ to $p_2^{-1}[U]$ such that $p\left(h_1^{-1}(u,a)\right) = h_2^{-1}\left(q(u,a)\right)$. By continuity, for each $a \in A$ the map q sends the connected subset $U \times \{a\} \subset U \times A$ into some connected component $U \times \{b(a)\} \subset U \times B$, and q is onto because p is onto. Since the composites

$$U \times \{a\} \;\longrightarrow\; p_1^{-1}[U] \;\longrightarrow\; U, \quad U \times \{b\} \;\longrightarrow\; p_2^{-1}[U] \;\longrightarrow\; U$$

are homeomorphisms, it follows that q must map $U \times \{a\}$ homeomorphically onto $U \times \{b(a)\}$; denote the corresponding homeomorphism from U to itself by q_a. If we compose h with the union of the homeomorphisms $\varphi = \cup_a\, q_a : U \times A \to U \times A$ and replace h with the composite $\varphi \circ h$, then we obtain a new map q' analogous to q but satisfying the condition that q maps $U \times \{a\}$ to $U \times \{b(a)\}$ to the identity; note that q' is onto because q is onto. It follows that q' is a covering space projection, and by part (i) of Exercise 1 it also follows that the map

$$p_1^{-1}[U] = p^{-1}\left[p_2^{-1}[U]\right] \;\longrightarrow\; p_2^{-1}[U]$$

determined by p is also a covering space projection.

If we apply the reasoning of the preceding paragraph to the open covering of E_2 by the sets $p_2^{-1}[U_\gamma]$, where U_γ belongs to the previously described open covering $\mathcal{U}$, we see that each of the maps

$$p_1^{-1}[U_\gamma] = p^{-1}\left[p_2^{-1}[U_\gamma]\right] \longrightarrow p_2^{-1}[U_\gamma]$$

is a covering space projection, and as noted above this suffices to prove that p itself is a covering space projection.

Note. Here is an example where p is not onto: Let X be an arbitrary nonempty space. Take $p : X \to X \amalg X$ to be inclusion into the first summand, and take $p_2 : X \amalg X \to X$ to be the map which is the identity on each summand. There is a variation of this exercise in which one replaces the surjectivity hypothesis on p by an assumption that E_2 is connected; the relationship with Exercise 2 is that if E_2 is connected then one can prove that p is onto.

3. (*i*) Let $y \in Y$, let U be an evenly covered open neighborhood of $f(x)$, and let V be an open neighborhood of x such that $f[V] \subset U$. We claim that V is evenly covered in $Y \times_X E$. The definitions imply that the inverse image of V is the subspace of $V \times E$ defined by $f(v) = p(e)$, and the inverse image is also the subspace of $V \times p^{-1}[U]$ defined by the same equation. If h is the homeomorphism $p^{-1}[U] \to U \times A$ which exists by the assumption that U is evenly covered, then under this homeomorphism the inverse image of V corresponds to the set of all points (v, u, a) in $V \times U \times A$ such that $f(v) = u$; *i.e*, the inverse image is given by the product of A with the graph of $f|U$. Since the graph is homeomorphic to U, it follows that the inverse image is homeomorphic to U via coordinate projection; one can check directly that all these maps are compatible with the appropriate projections onto U, and this shows that U is evenly covered.

For the second part, if a lifting φ exists, then the image of the map sending y to $(y, \varphi(y))$ is contained in $Y \times_X E$ and hence determines a map $s : Y \to Y \times_X E$; by definition the composite of s with projection onto Y is the identity. Conversely, if there is a map $s : Y \to Y \times_X E$ such that $p_{(Y,f)} s = 1_Y$, then the composite of s with $j : Y \times_X E \subset E$ satisfies $p \circ j \circ s = f$, and therefore $j \circ s$ is a lifting of f.■

(*ii*) If f is a subspace inclusion then $Y \times_X E$ is just the set of all points (y, e) in $Y \times p^{-1}[Y]$ such that $p(e) = y$. This maps homeomorphically to $p^{-1}[Y]$ by projection onto the second factor, and an explicit inverse is given by the map sending e to $(p(e), e)$. Both of these maps are compatible with respect to the various projections onto Y.■

4. Let $x \in X$ and let U be an evenly covered open neighborhood of x. Since X is totally disconnected there is an open subneighborhood $V \subset U$ such that V is open and closed in X. By continuity $p^{-1}[V]$ is open and closed in E.

If $y \in Y$, let $p(y) = x$, so that x has an evenly covered open neighborhood V which is also closed in X. Now let W be the sheet over V which contains y. We claim that W is open and closed in Y. Openness follows because W is a sheet over an evenly covered open subset of X. To prove that W is closed, let W' be the union of all the other sheets over V, so that W' is open and $E - W'$ is closed. Then $W = p^{-1}[V] \cap (X - W')$ shows that W is an intersection of two closed subsets and hence W is closed in Y.■

5. (*i*) More generally, if $\mathcal{B}$ is a base for the topology of X and $E \to X$ is a covering space projection, then the subset $\mathcal{B}'$ of all evenly covered open subsets of $\mathcal{B}$ is also a base because every evenly covered open set is also a union of basic open subsets. Therefore if X is second countable, then there is a countable base $\mathcal{B}$ of X by evenly covered open subsets.

Let $\mathcal{A}$ be the family of all open subsets $W \subset E$ which are sheets over evenly covered open subsets $V \subset X$ such that $V \in \mathcal{B}$. We claim that $\mathcal{A}$ is a base for the topology on E. Since the number of sheets over each V is countable by the countability assumption and the number of open sets in $\mathcal{B}$ is countable, it follows that $\mathcal{A}$ is a countable family. Therefore we need only show that $\mathcal{A}$ is a base for the topology.

Let $\mathcal{C}$ be the family of open subsets in E such that $W \in \mathcal{C}$ if and only if W is a sheet over an evenly covered open subset $V \subset X$. Then $\mathcal{C}$ is a base for the topology on E, so it will suffice to show that every subset in $\mathcal{C}$ is a union of open subsets in $\mathcal{A}$. Let $W \in \mathcal{C}$ as above, and assume it is a sheet over V. Then we can write $V = \cup_j U_j$ as a countable union of open sets $U_j \in \mathcal{B}$, and it follows that W is the union of the open subsets $N_j = p^{-1}[U_j] \cap W$. By definition the sets N_j belong to $\mathcal{A}$, so we have shown that $\mathcal{A}$ forms a base for the topology of E.■

(*ii*) Follow the hint. By a previously proved exercise in Munkres we know that E is $\mathbf{T}_3$, and by the first part of this exercise it is also second countable. Therefore the Urysohn Metrization Theorem implies that E is metrizable.■

VIII.5 : Simply connected spaces

Additional exercises

1. (*i*) We need to show that $g \cdot x = x$ for some x only if $g = 1$. Let (u, v) denote a point of $S^2 \times S^3$, and suppose that $g \cdot (u, v) = (u, v)$; *i.e.*, $g \cdot u = u$ and $g \cdot v = v$. If $g \in C_n$ then $g \cdot (u, v) = (u, g \cdot v)$, and the second coordinate is equal to v if and only if $g = 1$. On the other hand, if $g \notin C_n$ then $g \cdot (u, v) = (-u, v) \neq (u, v)$, so that $g \cdot (u, v) = (u, v)$ implies that g must be equal to 1.

(*ii*) The preceding result yields a free action of $D(2n)$ on the simply connected space $S^2 \times S^3$, and by the results in this section of the notes the fundamental group of the quotient space $X_n = S^2 \times S^3 / D(2n)$ has a fundamental group isomorphic to $D(2n)$.■

(*iii*) Let $\Delta : D(2n) \to \mathbb{Z}_2$ be the homomorphism defined in the exercise, and identify $\mathbb{Z}_2$ with ± 1 as usual. Then the coordinate projection $p : S^2 \times S^3 \to S^2$ has the property that $p(g \cdot (u, v)\,) = (\Delta(g) \cdot u)$, and this implies that p passes to a map q from X_n to $\mathbb{RP}^2$. Let γ be a great circle curve in S^2 which joins a basepoint z to its antipodal point $-z$, let $v_0 \in S^3$, take $\beta(t) = (\gamma(t), v_0) \in S^2 \times S^3$, and let $\rho : S^2 \times S^3 \to X$ be the quotient projection. Then $\rho \circ \gamma$ is a closed durve in X, and its image $q \circ \rho \circ \gamma$ generates the fundamental group of $\mathbb{RP}^2$. Therefore the map from $\pi_1(X)$ to $\pi_1()$ is nontrivial. Further analysis would show that the kernel of this map in fundamental groups is just C_n, but this was not asked for in the exercise.■

2. If the inclusion $S^1 \subset \mathbb{RP}^n$ were a retract, then it would induce a monomorphism of fundamental groups. Since $\pi_1(S^1) \cong \mathbb{Z}$ and $\pi_1(\mathbb{RP}^n)$ is finite for $n \geq 2$, this cannot happen.■

3. (*i*) Under the given conditions, the result from this section imply that the map $p_*; \pi_1(X) \to \pi_1(\mathbb{RP}^n \times \mathbb{RP}^n) \cong \mathbb{Z}^2 \times \mathbb{Z}_2$ is injective and the number of sheets in the covering is the index of the image of p_*. If there is only one sheet, then this map must be a homeomorphism (it is 1–1, onto, continuous and open). Otherwise the number of sheets is the index of a subgroup of a group of order 4, and as such this number must be finite and even.■(*ii*) If E is a connected space covering space of X satisfying the default hypotheses and $\pi_1(X)$ is finite of odd order, then $\pi_1(E)$ is isomorphic to a subgroup of $\pi_1(X)$ as in the first part of this exercise, but now the index of the subgroup must divide the odd number $|\pi_1(X)|$ and hence the index, which equals the number of sheets in the covering, must be (finite and) odd. In particular, the index cannot be equal to 2.■

VIII.6 : Homotopy of paths and line integrals

Additional exercises

1. Recall the definition of the winding number integral: If $p(z)$ is never zero on a circle C_R of radius R about the origin and $\Gamma(p, R)$ be the closed curve given by $p(\exp(R \cdot 2\pi\, i\, t))$ for $0 \leq t \leq 1$, then the winding number integral is equal to

$$\int_{\Gamma(p,R)} \frac{x\, dy - y\, dx}{x^2 + y^2}$$

and its value is an integral multiple of 2π. By the results of this section, the integral factor is equal to the degree of the following composite

$$S^1 \xrightarrow{R\times} C_R \xrightarrow{p} \mathbb{C} - \{0\} \xrightarrow{\mathbf{u}} S^1$$

where $R\times$ denotes multiplication by R and $\mathbf{u}$ sends z to $|z|^{-1} \cdot z$.

The preceding sentence implies that if the winding number is not zero then the displayed map is not homotopic to a constant in $\mathbb{C} - \{0\}$ and hence that $\Gamma(p, R)$ does not extend to a continuous map from D^2 into $\mathbb{C} - \{0\}$. Now $p(r \cdot \mathbf{v})$ is a continuous extension of $\Gamma(p, R)$ to a map $D^2 \to \mathbb{C}$, and by the preceding sentence we know that its image cannot be contained in $\mathbb{C} - \{0\}$. Therefore its image must contain the point 0; in other words there must be some z_0 such that $|z| < R$ and $p(z_0) = 0$.■

2. The underlying idea is to show that the winding number is defined for $p + q$ and it is equal to the winding number of p, which by hypothesis is nonzero.

First of all, the condition $|q| < |p|$ for $|z| \leq R$ implies that $|p + q| \geq |p| - |q| > 0$ for $|z| = R$, and therefore the winding number of $p+q$ can be defined. To prove the winding numbers are equal, it is only necessary to show that $\Gamma(p, R)$ and $\Gamma(p + q, R)$ are homotopic as maps into $\mathbb{C} - \{0\}$. One obvious idea is to consider the straight line homotopy $p + t\, q$ where $0 \leq t \leq 1$ and show that its image lies in $\mathbb{C} = \{0\}$. The verification of this statement is a slight embellishment on what was already shown: $|p + tq| \geq |p| - t|q| \geq |p| - |q| > 0$.■

3. We know that the value of the integral only depends upon the free homotopy class of γ, and since there are only countably many free homotopy classes in $[S^1, U]$ it follows that there are only countably many possible values for the integral. Furthermore, since the line integral of a concatenated curve is the sum of the line integrals of the two pieces, it follows that the line integral of f defines a homomorphism $S(f)$ from $\pi_1(U)$ to the additive complex numbers, with the value of $S(f)$ only depending upon the image of a class in $\pi_1(U)$ in $[S^1, U]$.

If the line integral has only finitely many values, then $S(f)$ has a finite image, and since $S(f)$ is a homomorphism it follows that every element in the image has finite order. However, the only elment of finite additive order in $\mathbb{C}$ is 0, and therefore it follows that the image of $S(f)$ must be $\{0\}$ and there is only one possible value for the line integral.■

SOLUTIONS TO EXERCISES FOR

MATHEMATICS 205A — Part 9

Fall 2014

IX. Computing fundamental groups

IX.1 : Free groups

Additional exercises

1. For the sake of definiteness, assume that x and y are free generators for F. If H has index 2 in F, then H is a normal subgroup and $F/H \cong \mathbb{Z}_2$. Therefore every such subgroup is the kernel of a surjection from F to $\mathbb{Z}_2$. CLAIM 1: This is a 1–1 correspondence.

We can check this directly. There are three nontrivial homomorphisms from F into $\mathbb{Z}_2$, and they are completely determined by the values of a homomorphism on the generators. Consider the effect of each example on the set $S = \{x, xy, y\}$. For the homomorphism sending (x, y) to $(1, 0)$ the intersection of S with the kernel is $\{x\}$, for the homomorphism sending (x, y) to $(0, 1)$ the the intersection of S with the kernel is $\{y\}$, and for the homomorphism sending (x, y) to $(1, 1)$ the the intersection of S with the kernel is empty.

Therefore there are exactly three subgroups of index 2 in F, one of which is normally generated by x and y^2, another of which is normally generated by y and x^2, and the last of which is normally generated by $x^{-1}y$ and x^2 (or $y^{-1}x$ and x^2, or xy and x^2, *etc.*).■

2. We shall verify that F/H satisfies the appropriate Universal Mapping Property with free generators corresponding to the elements of $X - Y$. Let G be a group, let $j : X \to F$ identify X with a subset of F, and let $h : X - Y \to G$ be a map of sets. Extend h to a mapping $h_0 : X \to Y$ by setting $h(y) = 1$ if $y \in Y$. Since F is free on X, there is a homomorphism $\eta_0 : F \to G$ such that $\eta_0 \circ j(x) = h_0(x)$ for all $x \in X$. Since $y \in Y$ implies $h_0(y) = 1$, the kernel of η_0 contains the normal subgroup H, and if $\pi : F \to F/H$ is the quotient map then there is a unique homomorphism $\eta : F/H \to G$ such that $\eta \circ \pi = \eta_0$.

Before proving that F/H has the Universal Mapping Property for $X - Y$, we need to check that the map $\pi \circ j : X \to F/H$ is 1–1. Perhaps the easiest way to do this is to let A be the free group on $X - Y$ and take the homomorphism $F \to A$ which sends $X - Y$ to these free generators and sends Y to $\{1\}$. Since the map $X - Y \to F \to F/H \to A$ is 1–1, it follows that $X - Y \to F/H$ is also 1–1, confirming what we expected.

To complete the proof of the Universal Mapping Property, identify $X - Y$ with a subset of F/H via the map sending x to $j'(x) = \pi \circ j(x)$, so that $\eta \circ j'(x) = \eta_0 \circ j(x) = h_0(x)$ for all $x \in X - Y$. Therefore F/H has the universal mapping property for $X - Y$.■

3. (i) Let $F_n' \subset F_n$ be the commutator subgroup; then the quotient is isomorphic to the free abelian group $A_n \cong \mathbb{Z}^n$ on n generators. If $T : F_n \to F_n$ is an automorphism and $h : F_n \to A_n$ is the quotient projection as in the statement of the exercise, then the kernel of $h \circ T$ must contain the commutator subgroup $F_n' = [F_n, F_n]$ because the image of the homomorphism is abelian. Therefore there is a unique homomorphism $\theta(T) : A_n \to A_n$ such that $h \circ T = \theta(T) \circ h$.

Suppose now that we are given two automorphisms T_1 and T_2. By the uniqueness statement proved in the preceding paragraph, it is enough to show that $h \circ T_1 \circ T_2 = \theta(T_1) \circ \theta(T_2) \circ h$. This follows because $h \circ T_1 \circ T_2 = \alpha(T_1) \circ h \circ T_2 = \alpha(T_1) \circ \alpha(T_2) \circ h$.■

(*ii*) The statement in the hint is true because $h \circ \mathrm{id}_{F_n} = h = \mathrm{id}_{A_n} \circ h$. Therefore if $S = T^{-1}$ we have

$$\mathrm{id}_{A_n} \quad = \quad \theta(\mathrm{id}_{F_n}) \quad = \quad \theta(S \circ T) \quad = \quad \theta(S) \circ \theta(T)$$

and by interchanging the roles of S and T we also have $\mathrm{id}_{A_n} = \theta(T) \circ \theta(S)$. Therefore $\theta(T)$ is an automorphism, and predictably its inverse is $\theta(S)$.■

(*iii*) Again for definiteness, let x and y denote the generators of F_2 which project down to the elements $(1, 0)$ and $(0, 1)$ in $A_2 \cong \mathbb{Z}^2$. Following the hint, we shall find automorphisms of F_2 which induce θ on A_2 for choices of θ corresponding to each one of the three given generators. To avoid space-consuming displays of 2×2 matrices we shall refer to the displayed matrices, in order from left to right, as the *diagonal generator*, the *transposition generator*, and the *shear generator*. For the diagonal generator, take the self-homomorphism T of F_2 which sends x to x^{-1} and y to itself; such a homomorphism exists because F_2 is free, and it is an automorphism because $T \circ T = \mathrm{id}$ (it is only necessary to check this on the free generators), so that T is equal to its own inverse. For the transposition generator, take the self-homomorphism T which interchanges x and y; once again $T \circ T = \mathrm{id}$ implies that T is its own inverse. Finally, for the shear generator, take the homomorphism T sending x to itself and y to xy. For this example we claim that the inverse is the homomorphism sending x to itself and y to $x^{-1}y$. Once again, to prove that $S \circ T$ and $T \circ S$ are the identities, it is enough to do so on the standard set of free generators. Clearly we have $S \circ T(x) = x = T \circ S(x)$ since $S(x) = T(x) = x$, and we also have

$$S \circ T(y) \; = \; S(xy) \; = \; S(x)S(y) \; = \; x \cdot (x^{-1}y) \; = \; y$$

$$T \circ S(y) \; = \; T(x^{-1}y) \; = \; T(x^{-1})T(y) \; = \; x^{-1} \cdot (xy) \; = y$$

and therefore we know that $S = T^{-1}$.■

4. (*i*) Take the map from F_{n-1} to G with sends the free generator $x_i \in F_{n-1}$ to $g_i \in G$. The extension of this map to a homomorphism is onto, and therefore G is isomorphic to a quotient of F_{n-1}.■

(*ii*) In any group G, if $g = g^1$ then either $g = 1$ or else $g^2 = 1$. The latter cannot happen in an odd order group unless $G = 1$, so this means that the nontrivial elements of G can be decomposed into $\frac{1}{2}(|G| - 1)$ pairs of the form $\{g_i, h_i = g_i^{-1}\}$, where $1 \leq i \leq k$ and $|G| = 2k + 1$.

In this case take the map from F_k to G with sends the free generator $x_i \in F_k$ to $g_i \in G$. The extension of this map to a homomorphism is onto, and therefore G is isomorphic to a quotient of F_k.■

IX.2 : Sums and pushouts of groups

Problems from Munkres, § 68, p. 421

2. (*a*) Let $1 \neq x_i \in G_i$ for $i = 1, 2$; then $x_1 x_2 x_1^{-1} x_2^{-1}$ is a reduced word, and by Step 4 in the proof of Munkres, Theorem 68.2 we know that this element is not the identity in $G_1 * G_2$. But this means that $x_1 x_2 \neq x_2 x_1$ whenever x_1 and x_2 are nontrivial elements of G_1 and G_2 respectively.■

(b) If x is a reduced word of even length, write it in the form $a_1b_1 \; \cdots \; a_kb_k$ where each a_j lies in one of the groups G_i and each b_j lies in the other group. It follows that for each $n > 0$ that x^n corresponds to the reduced word $a_1b_1 \; \cdots \; a_{nk}b_{nk}$ where the sequences satisfy the periodicity conditions $a_j = a_{j+k}$ and $b_j = b_{j+k}$ for $j \leq nk - k$. Since this is also a nontrivial word, it follows from the same reasoning as before that $x^n \neq 1$ in the free product. Therefore x has infinite order.

Suppose now that we have a reduced word x of odd length ≥ 3 (*this was* not *part of the problem as stated in Munkres, but clearly it is indispensable because a reduced word of length* 1 *cannot be conjugate to anything shorter*). In analogy with the preceding paragraph, write x in the form $b_0a_1b_1 \; \cdots \; a_kb_k$ where a_j and b_j are as before. We can easily find a shorter word which is conjugate to the given one because $b_0^{-1}xb_0$ is equal to $a_1b_1 \; \cdots \; a_k(b_kb_0)$. There are now two possibilities. If $b_kb_0 \neq 1$, then we have shown that x is conjugate to an element corresponding to a reduced word of even length $2k$. If $b_kb_0 = 1$, then we have shown that x is conjugate to an element corresponding to a reduced word of odd length $2k - 1$.■

(c) By part (b) and induction, every nontrivial word is either conjugate to a word whose length is either an even number or 1 (look at the shortest word in the conjugacy class, and note that a nontrivial word cannot be conjugate to the empty word). If the word x is conjugate to a word y of even length, then the orders of x and y are equal, and since y has infinite order it follows that the same holds for x. On the other hand, if x is conjugate to a word y of length 1, we know that y must correspond to a nontrivial element of G_1 or G_2, and if x has finite order then y must also have the same finite order.■

3. The easiest way to solve this exercise might be to look at the images of everything in the direct product $G_1 \times G_2$. The Universal Mapping Property for free products guarantees the existence of a homomorphism $\theta : G_1 * G_2 \to G_1 \times G_2$ such that $\theta \circ i_1(a) = (a, 1)$ and $\theta \circ i_2(b) = (1, b)$, where i_t denotes the standard injection of G_t into $G_1 \cap G_2$. The problem does not require a proof that cG_1c^{-1} is a subgroup, but this follows quickly from the fact that the latter is the image of G_1 under the conjugation automorphism of $G_1 * G_2$ sending x to cxc^{-1}.

Suppose that $a \in G_1$ is such that $cac^{-1} \in G_2$ It then follows that $\theta(a) \in G_1 \times \{1\}$ and $\theta(c)\theta(a)\theta(c)^{-1} \in \{1\} \times G_2$. CLAIM: $\theta(c)\theta(a)\theta(c)^{-1} \in G_1 \times \{1\}$, and this element corresponds to a conjugate of a in G_1. — If this is true, then $\theta(c)\theta(a)\theta(c)^{-1}$ belongs to $(G_1 \times \{1\}) \cap (\{1\} \times G_2)$, which is the trivial group, and furthermore a is conjugate to this element in G_1. In particular, a is conjugate in G_1 to the trivial element, and this implies that $a = 1$. To summarize, the claim implies that if $cac^{-1} \in G_2$ then $a = 1$ and therefore also $cac^{-1} = 1$.

To prove the assertions regarding $\theta(c)\theta(a)\theta(c)^{-1}$, write $c = u_1v_1 \; \cdots \; u_kv_k$ where $u_j \in G_1 \times \{1\}$ and $v_j \in \{1\} \times G_2$. If $c \neq 1$ we can do this using either a reduced word of even length or taking a reduced word of odd length and setting $u_1 = 1$ (if the word starts and ends with something from G_2) or $v_k = 1$ (if the word starts and ends with something from G_1). Since the images of G_1 and G_2 commute with each other, an inductive argument shows that

$$\theta(c)\,\theta(a)\theta(c)^{-1} \quad = \quad \theta(u_1v_1 \; \cdots \; u_kv_k)\,\theta(a)\theta(u_1v_1 \; \cdots \; u_kv_k)^{-1} \quad =$$

$$\theta(u_1v_1 \; \cdots \; u_{k-1}v_{k-1})\,\theta(u_kau_k^{-1})\theta(u_1v_1 \; \cdots \; u_{k-1}v_{k-1})^{-1} \quad = \quad \cdots \quad = \quad \theta(u_1 \; \cdots \; u_k\,a\,u_k^{-1} \; \cdots \; u_k^{-1})$$

where the expression in the last term is an element of G_1 which is conjugate (in G_1) to a. This is the claim in the preceding paragraph.■

1. To simplify the notation, if H is a group, then $\mathbf{Ab}(H)$ will denote the quotient $H/[H,H]$, where $[H,H]$ is the commutator subgroup (which is normal in H). We shall also denote $G_1 * G_2$ by G as in the statement of the exercise.

Starting with the abelinization homomorphisms $\alpha_i : G_i \to \mathbf{Ab}(G_i)$, we can define a homomorphism $\theta : G \to \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$ whose restriction to $G_1 \subset G$ is the map sending a to $(a,0)$ and whose restriction to $G_2 \subset G$ is the map sending b to $(0,b)$. By construction θ is onto, and since the codomain is an abelian group the kernel of θ must contain the commutator subgroup. Therefore θ factors as a composite $G \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$, where the first arrow is abelianization and the second will be denoted by φ.

For the same general reasons, the composites $G_i \to G = G_1 * G_2 \to \mathbf{Ab}(G)$ have factorizations $G_i \to \mathbf{Ab}(G_i) \to \mathbf{Ab}(G)$, and the induced maps of abelianizations will be denoted by J_i. Therefore we can define a homomorphism

$$\psi : \mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2) \;\longrightarrow\; \mathbf{Ab}(G)$$

such that $\psi(u,v) = J_1(u) + J_2(v)$. By construction the composites

$$G_i \to \mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_i) \quad = \quad G_i \to G \to G_i \to \mathbf{Ab}(G_i)$$

are the abelianization mappings, and therefore the composites $\mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_i)$ are identity mappings. Similarly, if $i \neq j$ then the triviality of the composites $G_i \to G \to G_j$ implies that the abelianized mappings $\mathbf{Ab}(G_i) \to \mathbf{Ab}(G) \to \mathbf{Ab}(G_j)$ are zero homomorphisms. If we combine these with the definitions of φ and ψ, we see that $\varphi \circ \psi$ is the identity on $\mathbf{Ab}(G_1) \oplus \mathbf{Ab}(G_2)$. We claim these maps are isomorphisms, and to prove this it will suffice to show that ψ is onto. However, this follows quickly because we know that G is generated by the images of G_1 and G_2, which implies that $\mathbf{Ab}(G)$ is generated by the images of $\mathbf{Ab}(G_1)$ and $\mathbf{Ab}(G_2)$.■

3. For the sake of definiteness, we shall assume that $m \geq n$ (it will be clear that the case $m \leq n$ can be handled similarly).

(a) If G_1 and G_2 are abelian groups, then Exercise 1 implies that $\mathbf{Ab}(G_1 * G_2) \cong G_1 \oplus G_2$. If we specialize to the case where $G_1 = \mathbb{Z}_m$ and $G_2 = \mathbb{Z}_n$, this implies that $\mathbf{Ab}(G_1 * G_2)$ is a finite group of order mn.■

(b) Follow the hint. By Exercise 68.2 in Munkres, the only elements of finite order in $G_1 * G_2$ are those which are conjugate to elements in either G_1 or G_2, and thus if $g \in \mathbb{Z}_m * \mathbb{Z}_n$ has finite order, this order must divide either m or n. Since we are assuming that $m \geq n$, the largest possible order is m, and in fact this order is realized by the generator of $\mathbb{Z}_m$.■

(c) If $G = \mathbb{Z}_m * \mathbb{Z}_n$, then $|\mathbf{Ab}(G)| = mn$ implies that mn is uniquely determined by G, and by (b) we know that m is uniquely determined by G. Therefore $n = (mn)/m$ is also uniquely determined by G.

4. The goal of the problem is to find finite abelian groups G_1, G_2, H_1, H_2 such that $|G_1| \neq |H_1|$ and $|G_2| \neq |H_2|$ such that $G_1 \times G_2 \cong H_1 \times H_2$, and the hint is to use the abstract version of the Chinese Remainder Theorem: $\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_{ab}$ *if a and b are relatively prime.* — The latter can be found in nearly every upper level undergraduate textbook on abstract algebra or elementary number theory, so we shall not prove it here.

Since $2, 3, 5$ are pairwise relatively prime the Chinese Remainder Theorem and $(2 \cdot 3) \cdot 5 = 30 = 2 \cdot (3 \cdot 5)$ imply that

$$\mathbb{Z}_{30} \cong \mathbb{Z}_6 \times \mathbb{Z}_5 \ , \qquad \mathbb{Z}_{30} \cong \mathbb{Z}_2 \times \mathbb{Z}_{15}$$

so we get the desired examples if we take $G_1 = \mathbb{Z}_6$, $G_2 = \mathbb{Z}_5$, $H_1 = \mathbb{Z}_2$, and $H_2 = \mathbb{Z}_{15}$.■

Additional exercises

1. We shall repeatedly use the fact that a free product of two groups $*_i\ L_i$ is uniquely characterized up to isomorphism by the fact that homomorphisms from $*_i\ L_i$ to another group M correspond bijectively to homomorphisms from the summands L_i into M, and the correspondence is given by restricting to the subgroups L_i.

For $(G * H) * K$, the preceding paragraph means that homomorphisms from this group to some other group M are in 1–1 correspondence with homomorphisms from $G * H$ and K into M. However, homomorphisms from $G * H$ into M in 1–1 correspondence with homomorphisms from G into M and from H into M. Combining these, we see that homomorphisms from $(G * H) * K$ into M are in 1–1 correspondence with homomorphisms $G \to M$, $H \to M$ and $K \to M$. Since this is the defining condition for a free product of the three groups G, H and K it follows that $(G * H) * K$ is in fact a free product of these three groups. Similarly, homomorphisms from $G * (H * K)$ into some other group M are in 1–1 correspondence with homomorphisms from G and $H * K$ into M, and since homomorphisms from $H * K$ into M in 1–1 correspondence with homomorphisms from H into M and from K into M, we see that homomorphisms from $G * (H * K)$ into M are in 1–1 correspondence with homomorphisms $G \to M$, $H \to M$ and $K \to M$; as before, this means that $G * (H * K)$ is in fact a free product of G, H and K.

Finally, homomorphisms from both $G * H$ and $H * G$ into an arbitrary group H correspond bijectively to homomorphisms $G \to M$ and $H \to M$, and this yields an isomorphism between $G * H$ and $H * G$.■

2. (i) Follow the hint and note that $K * K \cong K$ because $K * K$ is a free group on $\aleph_0 + \aleph_0 = \aleph_0$ generators if K is a free group on $\aleph_0$ generators. If H_1 is finite but nontrivial and $H_2 = H_1 * K$, then H_1 is finite but H_2 is infinite. On the other hand, we have

$$H_2 * K \cong (H_1 * K) * K \cong H_1 * (K * K) \cong H_1 * K$$

which is what we wanted to prove.■

(ii) The underlying ideas are the same, but here we have $K \times K = K \oplus K$ is isomorphic to K because $K \oplus K$ is a free abelian group on $\aleph_0$ generators if K is. If H_1 is finite but nontrivial and $H_2 = H_1 * \times K$, then H_1 is finite but H_2 is infinite. On the other hand, we have

$$H_2 \times K \cong (H_1 \times K) \times K \cong H_1 \times (K \times K) \cong H_1 \times K$$

which is what we wanted to prove.■

(iii) Once again, this is the same basic idea, but now we are working with topological spaces. An explicit isometry from $Y \times Y$ to Y is given as follows: Let $\{\mathbf{e}_j\}$ denote the set of standard unit vectors in Y, and take the linear isomorphism from $Y \times Y$ to Y which sends $(\mathbf{e}_j, \mathbf{0})$ to $\mathbf{e}_{2j-1}$ and $(\mathbf{0}, \mathbf{e}_j)$ to $\mathbf{e}_{2j-1}$. This is clealy an invertible linear transformation, and if one imposes the metrics associated to the usual dot products (so that the unit vectors are orthonormal and $Y \times \{\mathbf{0}\}$ is orthogonal to $\{0\} \times Y$), then this linear isomorphism is an isometry of inner product spaces, which implies among other things that $Y \times Y$ is homeomorphic to Y.

Let X_1 be a compact metric space, and let $X_2 = X_1 \times Y$. Then X_1 is not homeomorphic to X_2 but $X_1 \times Y$ is homeomorphic to $X_2 \times Y$ because we have $X_2 \times Y = X_1 \times Y \times Y \cong X_1 \times Y$.■

3. As before the proof that $h \times h$ is the identity reduces to showing that $h \times h$ maps the free generators x and y to them selves. Since $h(x) = y$ and $h(y) = x$, this follows immediately.

Suppose now that $h(w) = w$ for some nontrivial reduced word w in x and y. If w begins with a power of x then $h(w)$ begins with a power of y and vice versa. By the unique factorization property for nontrivial reduced words, it follows that $h(w)$ cannot be equal to w.■

Note. In contrast, the induced automorphism $\theta(h)$ of $\mathbb{Z}^2$ sends $(1,1)$ to itself.

IX.3 : The Seifert – van Kampen Theorem

Problems from Munkres, § 70*, p.* 433

1. By the Seifert-van Kampen Theorem it will suffice to show that

$$\begin{array}{ccc} \pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U,p) \\ \Big\downarrow i_{2*} & & \Big\downarrow \\ \pi_1(V,p) & \longrightarrow & \pi_1(U,p)/N_1 \ * \ \pi_1(V,p)/N_2 \end{array}$$

is a pushout diagram if i_{1*} and i_{2*} are trivial homomorphisms, where the maps from $\pi_1(U,p)$ and $\pi_1(V,p)$ to the respective quotients $\pi_1(U,p)/N_1$ and $\pi_1(V,p)/N_2$ followed by the usual injections J_1 and J_2 into the free product. Denote the quotient projections $\pi_1(U,p) \to \pi_1(U,p)/N_1$ and $\pi_1(V,p) \to \pi_1(V,p)/N_2$ by q_1 and q_2 respectively.

Suppose that we are given homomorphisms $A : \pi_1(U,p) \to G$ and $B : \pi_1(V,p) \to G$ such that $A \circ i_{1*} = B \circ i_{2*}$. Since i_{1*} and i_{2*} are trivial it follows that we have factorizations through the respective quotients; *i.e.*, we have $A = A' \circ q_1$ and $B = B' \circ q_2$ for uniquely determined homomorphisms A' and B'. By the Universal Mapping Property for free products, there is a unique homomorphism C from the free product into G whose restrictions to $\circ J_1$ and $C \circ J_2$ to $\pi_1(U,p)/N_1$ and $\pi_1(V,p)/N_2$ are equal to A' and B' respectively, and therefore we also have $C \circ (J_1 \circ q_1) = A' \circ q_1 = A$ and $C \circ (J_2 \circ q_2) = B' \circ q_2 = B$. To complete the proof, we need to show that if $D : \pi_1(U,p)/N_1 * \pi_1(V,p)/N_2$ is an arbitrary homomorphism such that $D \circ (J_1 \circ q_1) = A$ and $D \circ (J_2 \circ q_2) B$, then $D = C$. If D satisfies these conditions then we have $D \circ J_1 \circ q_1 = C \circ J_1 \circ q_1$ and $D \circ J_2 \circ q_2 = C \circ J_2 \circ q_2$; since q_1 and q_2 are onto, the given equations imply that $D \circ J_1 = C \circ J_1$ and $D \circ J_2 = C \circ J_2$. We can now use the uniqueness condition in the Universal Mapping Property for free products to conclude that $D = C$. This completes the proof that the diagram at the beginning of this solution is a pushout.■

3. (*a*) If G_1 has a finite generator set X_1 with a finite relation set R_1 and G_2 has a finite generator set X_2 with a finite relation set R_2, then $G_1 * G_2$ has a finite generator set $X_1 \amalg X_2$ with a finite relation set $R_1 \amalg R_2$,■

(*b*) Follow the hint, but work more generally with a pushout

$$\begin{array}{ccc} K & \xrightarrow{i_1} & H_1 \\ \Big\downarrow i_2 & & \Big\downarrow j_1 \\ G_2 & \xrightarrow{j_2} & G \end{array}$$

where K is finitely generated and H_1 and H_2 are finitely presented.

The construction of pushouts in Section IX.2 shows that G is isomorphic to the quotient of $\Gamma = G_1 * G_2$ by the normal subgroup N which is generated by all elements of the form $i_1^{-1}(k)\, i_2(k)$, where k runs through all the elements of G. CLAIM: *If $k_1, \cdots, k_r$ generate K, then N is also the smallest normal subgroup containing the finite set $S = \{i_1^{-1}(k_t)\, i_2(k_t) \mid 1 \leq t \leq r\}$.*

If the claim is true, we can complete the solution as follows: By Exercise 1 we know that $G_1 * G_2$ is finitely presented, and by the claim we know that N is finitely normally generated, so if $G_1 * G_2$ is presented with finite generating set X and finite relation set R, then we obtain the quotient by expanding R to a set which also includes a finite family of words in the generators which map to the elements in the set S.

We now prove the claim. Let N_0 be the subgroup normally generated by S, so that $N_0 \subset N$. To prove the reverse inclusion, consider the map of quotient groups $\pi : \Gamma/N_0 \to \Gamma/N$ which sends each coset of N_0 to the coset N which contains it, and let $\rho : G \to G/N_0$ be the usual quotient space projection. By definition, N_0 is normally generated by the elements $i_1^{-1}(k_t)\, i_2(k_t)$, and therefore $\rho \circ i_1(k_t) = \rho \circ i_2(k_t)$ for all t. Since ρ, i_1 and i_2 are homomorphisms and the elements k_t generate K, it follows that $\rho \circ i_1(k) = \rho \circ i_2(k)$ for all $k \in K$. But this means that the normal subgroup N_0 contains all elements of the form $i_1^{-1}(k)\, i_2(k)$ where $k \in K$, and since these elements normally generate N it follows that all of N_0 is contained in N.■

Additional exercises

1. (*i*) For this part of the exercise, in the pushout diagram

$$\begin{array}{ccc} \pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U, p) \\ \downarrow i_{2*} & & \downarrow j_{1*} \\ \pi_1(V, p) & \xrightarrow{j_{2*}} & \pi_1(X, p) \end{array}$$

we know that $\pi_1(U, p)$ is trivial and $\pi_1(V, p)$ is abelian. It will suffice to prove that the map $\pi_1(V, p) \to \pi_1(X, p)$ is onto. We know that $\pi_1(X, p)$ is generated by the images of $\pi_1(U, p)$ and $\pi_1(V, p)$ and since the image of the first group must be trivial it follows that $\pi_1(V, p)$ generates $\pi_1(X, p)$, which means that $\pi_1(V, p) \to \pi_1(X, p)$ is onto.■

(*ii*) Let X be the Figure Eight Space which is a union of two closed subspaces $C_1 \cup C_2$ such that each is homeomorphic to S^1 and $C_1 \cap C_2$ consists only of the basepoint p. Choose points $q_i \in C_i - \{p\}$, and let U_1 and U_2 be $X - \{q_2\}$ and $X - \{q_1\}$ respectively (note the switch in subscripts — this is not a misprint). Then C_i is a strong deformation retract of U_i and $U_1 \cap U_2$ is contractible, so that the pushout diagram associated to $\pi_1(X = U_1 \cup U_2$ is given as follows:

$$\begin{array}{ccc} \{0\} \cong \pi_1(U \cap V, p) & \xrightarrow{i_{1*}} & \pi_1(U, p) \cong \mathbb{Z} \\ \downarrow i_{2*} & & \downarrow j_{1*} \\ \mathbb{Z} \cong \pi_1(V, p) & \xrightarrow{j_{2*}} & \pi_1(X, p) \cong F_2 \end{array}$$

In this example the fundamental groups of U and V are abelian but the fundamental group of X is not.■

2. (i) Once again we know that $\pi_1(X)$ is generated by the images of $\pi_1(U)$ and $\pi_1(V)$. Since $\pi_1(U \cap V)$ maps onto both of the latter groups, it follows that all the generators for $\pi_1(X)$ actually lift back to $\pi_1(U \cap V)$.■

(ii) As in (i) if A and B denote generating sets for $\pi_1(U)$ and $\pi_1(V)$ respectively and A' and B' denote their images in $\pi_1(X)$, then $A' \cup B'$ generates $\pi_1(X)$. But if A and B are finite, then so is $A' \cup B'$.■

3. We begin with a general statement. Suppose that we have a group G presented as a quotient F/N where F is freely generated by X and N is normally generated by relations $R \subset F$. Then $\mathbf{Ab}(G)$ is isomorphic to $F/N\,[F,F]$, which is isomorphic to

$$(F/[F,F])/(N\,[F,F]/[F,F])$$

in which $N\,[F,F]/[F,F]$ is the image of $N \subset F \to F/[F,F]$.

For the example in this exercise, the preceding observation shows that the abelianization $\mathbf{Ab}(G)$ is the quotient of $\mathbb{Z}^2$ modulo the subgroup generated by the single abelianized relation $(3,-2)$ because the abelianizations of the other two relations are trivial. One easy way of seeing that the quotient is infinite cyclic is to observe that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ sending (x,y) to $2x+3y$ is onto and its kernel is the cyclic subgroup generated by $(3,-2)$.■

(ii) Let $\rho : G \to G/N$ be the quotient group projection. Since $xy^{-1} \in N$ it follows that $\rho(x) = \rho(y)$. By definition we know that $x^3 = y^2$, so that $\rho(x)^3 = \rho(y)^2$. If we combine the preceding two sentences we find that $\rho(x)^3 = \rho(x)^2$, which means that $\rho(x) = 1$ and hence also that $\rho(y) = \rho(x) = 1$; *i.e.*, we have $x, y \in N$. Since x and y generate G, this means that $N = G$.■

4. By the Seifert-van Kampen Theorem, it will suffice to prove the following algebraic result about pushout diagrams of groups: If we are given an onto homomorphism $i_1 : K \to H_1$ and an isomorphism $i_2 : K \to H_2$, then the following square is a pushout diagram:

$$\begin{array}{ccc} K & \xrightarrow{i_1} & H_1 \\ \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle \mathrm{id}} \\ H_1 & \xrightarrow{i_1 i_2^{-1}} & H_1 \end{array}$$

This square commutes because both composites from K to H_1 are equal to i_1.

As usual, we shall prove the given square is a pushout by verifying that it has the Universal Mapping Property. So let $f_1 : H_1 \to M$ and $f_2 : H_2 \to M$ satisfy $f_1 \circ i_1 = f_2 \circ i_2$. We need to find a unique map $h : H_1 \to M$ such that $h \circ j_i = f_i$, where j_1 is the identity and $j_2 = i_1 \circ i_2^{-1}$. If we take $h = f_1$, then $h \circ j_1 = h \circ \mathrm{id} = f_1$ and $h \circ j_2 = h \circ i_1 \circ i_2^{-1} = f_1 \circ i_2^{-1} = f_2 \circ i_2 \circ i_2^{-1} = f_2$, so there is a map from H_1 to M with the right properties. To show that a map with the right properties is unique, note that if $k \circ j_i = f_i$ for $i = 1, 2$ then $k = k \circ \mathrm{id} = k \circ j_1 = f_1$, so that $k = h$.■

5. Consider the associated pushout diagram:

$$\begin{array}{ccc} \mathbb{Z} \times \mathbb{Z} & \xrightarrow{p_1} & \mathbb{Z} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle j_1} \\ \mathbb{Z} & \xrightarrow{j_2} & A \end{array}$$

Since p_1 and p_2 are onto, it follows that the composite $\mathbb{Z} \times \mathbb{Z} \to A$ is also onto (see Exercise 2); note that A is abelian because it is a homomorphic image of $\mathbb{Z} \times \mathbb{Z}$. Since $(1,0)$ and $(0,1)$ are in the kernels of p_2 and p_1 respectively, it follows that both these elements map to zero in A, and since the two elements in question generate $\mathbb{Z} \times \mathbb{Z}$, it follows that everything in $\mathbb{Z} \times \mathbb{Z}$ maps to zero in A. If we combine this with the conclusion of the previous paragraph, we see that A must be the trivial group.$\blacksquare$

6. Follow the hint; let $U = M - Y$ and $V = M - X$. Then U is homeomorphic to $X \amalg [0,1)$ with $x \in X$ identified to 0, and V is homeomorphic to $(0,1] \amalg Y$ with 1 identified to $y \in Y$. Both U and V are open in M, and their intersection is homeomorphic to the open interval $(0,1)$. Furthermore, X and Y are strong deformation retracts of U and V respectively. Therefore the Seifert-van Kampen Theorem implies that the fundamental group of M is the free product of the fundamental groups of X and Y (we have not been careful about the basepoints because the isomorphism type of the fundamental groups of the spaces in this exercise are isomorphic for all choices of basepoints).$\blacksquare$

IX.4 : Examples and computations

Additional exercises

1. (*i*) One can also model X topologically as the subspace $C \subset \mathbb{R}^3$ given by $S^2 \cup \{(0.0)\} \times [-1,1]$. We shall prove that this space is homeomorphic to the subspace of $\mathbb{R}^4$ described in the exercise as follows: Take the identity map on S^2, and map a point of the form $(0,0,t)$, where $-1 \leq t \leq 1$, to the point $(0,0,t,\sqrt{1-t^2})$. One can check directly that this map is continuous and 1–1 onto, so it is a homeomorphism because X is compact Hausdorff.$\blacksquare$

(*ii*) Follow the hint. The space which interests us is $S^2 \cup A$, and $D^3 \cup A$ is formed from it by regularly attaching a 3-cell, so by Proposition 3 IX.4.2 we know that $\pi_1(S^2 \cup A) \cong \pi_1(D^3 \cup A)$.

(*iii*) If $B \subset \mathbb{R}^3$ is the straight line segment described in the exercise which joins the north and south poles of S^2, then a retraction $D^3 \to B$ is defined by sending (x,y,z) to z; if we also take the straight line homotopy between these two points (which stays inside D^3 by convexity), we obtain deformation retract data for $B \subset D^3$. Now $A \cap B$ consists of the two points $\pm \mathbf{e}_3$, and by an exercise from Unit VII it follows that $B \cup A$ is a strong deformation retract of $D^3 \cup A$. Since $B \cup A$ is the union of two closed subspaces homeomorphic to $[-1,1]$ which meet at their endpoints, the space $A \cup B$ is homeomorphic to S^1; for the sake of completeness, we note that an explicit homeomorphism is given by sending one copy of $[-1,1]$ to the upper semicircle by the mapping $t \to (t, \sqrt{1-t^2})$ and sending the other copy of $[-1,1]$ to the lower semicircle by the mapping $t \to (t, -\sqrt{1-t^2})$.

Finally, the preceding observations combine to yield the fundamental group relationships $\pi_1(S^2 \cup A) \cong \pi_1(D^3 \cup A) \cong \pi_1(B \cup A) \cong \pi_1(S^1) \cong \mathbb{Z}$, as asserted in the statement of the exercise.$\blacksquare$

2. The intersection of $D^2 \times \{0\}$ with S^2 is equal to $S^1 \times \{0\}$, so if we take $A = S^2$ and $B = D^2$ then we have an example with the properties described in the discussion before the statement of Proposition IX.4.2. Therefore we can apply this result to conclude that the map of fundamental groups $\pi_1(S^2) \to \pi_1(X)$ is onto. Since $\pi_1(S^2)$ is trivial, it follows that $\pi_1(X)$ must also be trivial.$\blacksquare$

3. (*i*) Follow the hint. The data in the problem yield the following commutative diagram, in which the vertical arrows j_k are isomorphisms:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "j_1"] & \cdots \; c_k^{-1}\mathbb{Z} \arrow[r, "\subset"] \arrow[d, "j_k"] & c_{k+1}^{-1}\mathbb{Z} \arrow[r] \arrow[d, "j_{k+1}"] & \cdots \; \mathbb{Q} \\
A_1 \arrow[r] & \cdots \; A_k \arrow[r, "h_k"] & A_{k+1} \arrow[r] & \cdots \; G
\end{tikzcd}$$

If we define $\varphi_k : c_k^{-1}\mathbb{Z} \to G$ by $g_k \circ j_k$, then we have the recursive property

$$\varphi_{k+1}|c_k^{-1}\mathbb{Z} \quad = \quad g_{k+1} \circ j_{k+1}|c_k^{-1}\mathbb{Z} \quad = \quad g_{k+1} \circ h_k \circ j_k \quad = \quad g_k \circ j_k \quad = \quad \varphi_k$$

and therefore we can assemble these mappings to produce a homomorphism $\varphi : \mathbb{Q} \to G$. This map is onto, for each $a \in G$ has the form $g_k(b)$ for some k and $b \in A_k$, so if $j_k(b') = b$ we have $\varphi_k(b') = a$. In particular, this implies that G is abelian, so we shall use 0 to denote the neutral element in G. To see that φ is 1–1, suppose that $x \in \mathbb{Q}$ maps to 0, and choose k such that $x \in c_k^{-1}\mathbb{Z}$. Then $j_k(x) \in A_k$ maps to 0 in G, and therefore there is some $M \geq 0$ such that $j_k(x)$ maps to 0 in A_{k+M}. But the map $A_k \to A_{k+M}$ is equivalent to a nonzero map $\mathbb{Z} \to \mathbb{Z}$, so if $j_k(x)$ maps to 0 in some A_{k+M} then we must have $j_k(x) = 0$. Since j_k is an isomorphism we must have $x = 0$. Therefore $\varphi : \mathbb{Q} \to G$ is an isomorphism.■

(ii) The mapping $j[d]$ is 1–1 because the composite of $j[d]$ with projection onto the D^2 factor is the standard inclusion of S^1 in D^2, and the map $j[d]_*$ in fundamental groups corresponds to multiplication by d because the composite of $j[d]$ with projection onto the S^1 factor has degree d and this coordinate projection map induces an isomorphism $\pi_1(S^1 \times D^2) \to \pi_1(S^1)$.■

(iii) A more concrete approach to constructing E is to describe it as a subspace of $\mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$; more precisely, we shall realize each S_k as a subset of $\mathbb{C}^2 \times [k, k+1]$ such that the continuous mapping $S_k \to [k+1]$ corresponds to the last coordinate. Consider the subspace T_k of $\mathbb{C}^2 \times [k, k+1]$ consisting of $S^1 \times D^2 \times \{k+1\}$ together with the image of $S^1 \times [0,1]$ under the continuous mapping θ_k defined by

$$\theta_k(z, t) \quad = \quad (tz^{d_k} + (1-t)z,\, tz,\, t+k) \ .$$

We claim that θ_k is 1–1, and from this it follows that the standard quotient map from S_k to T_k is a homeomorphism onto its image. So suppose that $\theta_k(z,t) = \theta_k(z', t')$. Equating the third coordinates, we see that $t + k = t' + k$, so that $t = t'$. Now equating the second coordinates, we see that $tz = tz'$ so that either $t = 0$ (and hence $t' = 0$ or else $z = z'$; in the second case we are finished, so assume that $t = 0$ and look at the first coordinates. When $t = 0$ the first coordinate equation reduces to $z = z'$, so we have shown that $(z,t) = (z',t')$ must always hold. — Continuing, we see that the union $\cup_{j \leq k} T_k$ is homeomorphic to E_k, and if we set $T = \cup_k T_k$ we have a 1–1 onto continuous mapping $E \to T$. Projection onto the final coordinate in $\mathbb{R}^5 = \mathbb{C} \times \mathbb{C} \times \mathbb{R}$ yields a continuous mapping from T to $[0, \infty)$ such that the composite $E \to T \to [0, \infty)$ has all the right properties. Furthermore, this mapping sends the inverse image of $[0, k)$ homeomorphically to the inverse image of $[0, k)$ for all k, and from this one can prove that the map $E \to T$ is actually a homeomorphism (but this will not be needed to carry out the computations).

We now need to verify the assertion about the maps in fundamental groups associated to the inclusions $E_k \to E_{k+1}$. To start, we claim that for each k the inclusion $S^1 \times \{0\} \times \{k+1\} \subset S_k$ is a deformation retract. Since we know that the inclusion $S^1 \times \{0\} \times \{k+1\} \subset S^1 \times D^2 \times \{k+1\}$ is a deformation retract, it will suffice to show that $S^1 \times D^2 \times \{k+1\} \subset S_k$ is a deformation retract. This follows because $S_k = F_1 \cup F_1$, where $F_1 = S^1 \times D^2 \times \{k+1\}$ and F_2 is homeomorphic to $S^1 \times [0,1]$ such that $S^1 \times \{1\}$ corresponds to $F_1 \cap F_2$. We can now proceed by induction on j to show that the inclusion

$$S^1 \times \{0\} \times \{k+1\} \quad \subset \quad \bigcup_{i=j}^{k} E_i$$

is a deformation retract for $j = k, k-1, \cdots, 1$. Furthermore, it also follows that $S_{k+1} \subset E_{k+1}$ is a deformation retract.

The algebraic implication of the preceding paragraph is that the homomorphism $\pi_1(E_k) \to \pi_1(E_{k+1})$ is equivalent to the homomorphism $\pi_1(S^1 \times \{0\} \times \{k+1\}) \to \pi_1(S_{k+1})$ induced by inclusion. Since the composite of this inclusion with the retraction $E_{k+1} \to S^1 \times \{0\} \times \{k+2\}$ has degree d, it follows that all the homomorphisms of fundamental groups in this paragraph are equivalent to multiplication by d_k on $\mathbb{Z}$.$\blacksquare$

Note. Some of the mappings constructed in (iii) do not preserve basepoints particularly well, but this will not cause problems because in all cases the spaces are homotopy equivalent to S^1. This means that their fundamental groups are abelian and there are unique change of basepoint isomorphisms.

(iv) Each subset S_k is compact, and since E_k is a quotient of a finite union of subsets homeomorphic to $S_1, \cdots, S_k$, it follows that E_k is also compact. Furthermore, if $K \subset E$ is compact then its image in $[0, \infty)$ will also be compact, and since this image is contained in some closed interval $[0, M]$ it follows that $K \subset E_M$ for some M. The statements about the topology of E all follow from the fact that E is homeomorphic to T (but we shall not need these in the next step, which is the last one).$\blacksquare$

(v) We have shown that the diagram of fundamental group maps is the same as the algebraic diagram considered in (i), so by (i) it is only necessary to check that it satisfies properties (2) and (3) in (i). These follow from the Compact Supports Property for fundamental groups (Proposition VIII.1.12) and the fact that every compact subset of E is contained in some E_k.$\blacksquare$

(vi) Everything will go through if we modify the definition of the integer sequence d_k; specifically, if we are only interested in fractions which are monomials in S we can take d_k to be the product of the first k primes in S if $|S| \geq k$ and taking d_k to be the product of all the primes in S if $|S| < k$. If we now define c_k as before to be $d_1 \cdots d_{k-1}$ for the new sequence $\{d_k\}$, then $S^{-1}\mathbb{Z}$ will be the union of the sets $c_k^{-1}\mathbb{Z}$.$\blacksquare$