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Solutions to *Algebraic Topology*[*]

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Contents

1 The Fundamental Group

1.1 Basic Constructions

3. **Exercise.** For a path-connected space X, show that $\pi_1(X)$ is Abelian if and only if all basepoint-change homomorphisms β_h depend only on the endpoints of the path h.

   Solution. Suppose that for any two paths g and h from x_0 to x_1, the isomorphisms $\pi_1(X, x_0) \to \pi_1(X, x_1)$ given by $f \mapsto g^{-1}fg$ and $f \mapsto h^{-1}fh$ are the same. Now choose $f, f' \in \pi_1(X, x_0)$. We wish to show that $f'f = ff'$. Note that f' is homotopy equivalent to a composition gh^{-1}, where g and h are paths from x_0 to x_1, for the following reason. We can pick any point y on the path f' and let p be a path from y to x_1. Then the path from x_0 to y along f' composed with p is the desired g, and p^{-1} composed with the path from y to x_0 along f' is the desired

[*] by Allen Hatcher

h^{-1}. However, we know that $h^{-1}fh \simeq g^{-1}fg$, which can be rewritten as $gh^{-1}f \simeq fgh^{-1}$. Since f' is homotopic to gh^{-1}, this gives $f'f \simeq ff'$, so $\pi_1(X, x_0)$ is Abelian.

Conversely, suppose that $\pi_1(X, x_0)$ is Abelian and let g and h be two paths from x_0 to x_1. Then we get two isomorphisms $\pi_1(X, x_0) \to \pi_1(X, x_1)$ via $f \mapsto g^{-1}fg$ and $f \mapsto h^{-1}fh$, and we wish to show these two maps are the same. Note that hg^{-1} is a loop based at x_0, so is an element of $\pi_1(X, x_0)$. For any $f \in \pi_1(X, x_0)$, we have $hg^{-1}f \simeq fhg^{-1}$, which can be rewritten $g^{-1}fg \simeq h^{-1}fh$, so the two maps are indeed equal. □

6. **Exercise.** We can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(\mathrm{S}^1, s_0) \to (X, x_0)$. Let $[\mathrm{S}^1, X]$ be the set of homotopy classes of maps $\mathrm{S}^1 \to X$, with no conditions on basepoints. Thus there is a natural map $\Phi\colon \pi_1(X, x_0) \to [\mathrm{S}^1, X]$ obtained by ignoring basepoints. Show that Φ is onto if X is path-connected, and that $\Phi([f]) = \Phi([g])$ if and only if $[f]$ and $[g]$ are conjugate in $\pi_1(X, x_0)$. Hence Φ induces a one-to-one correspondence between $[\mathrm{S}^1, X]$ and the set of conjugacy classes in $\pi_1(X)$, when X is path-connected.

   Solution. Choose $f, g \in \pi_1(X, x_0)$. Ignoring the base point, we will show that fgf^{-1} is homotopic to g. Without loss of generality, we may assume that fgf^{-1} traverses f, g, and f^{-1} on the intervals $[0, 1/3]$, $[1/3, 2/3]$, and $[2/3, 1]$, respectively. Thinking of S^1 as $\mathbf{R}/\mathbf{Z}$, we can start at $1/3$ and end at $4/3$ (this corresponds to a free homotopy that moves the base point). This means that fgf^{-1} is free homotopic to $gf^{-1}f$, which is homotopic to g, so conjugacy classes map into homotopy classes of maps $\mathrm{S}^1 \to X$. Any homotopy class of maps $\mathrm{S}^1 \to X$ can be represented by some loop in X. Since X is path-connected, this can be extended to a loop based at x_0, and such a loop will be mapped by Φ to this homotopy class, so Φ is surjective.

   To see that Φ is injective, let $f, g \in \pi_1(X, x_0)$ be elements that are homotopic if we ignore base points (i.e., $\Phi(f) = \Phi(g)$). Then there is a continuous map $H\colon [0,1]^2 \to X$ such that $H(0,t) = H(1,t)$ for all t, and $H(t,0) = f(t)$ and $H(t,1) = g(t)$. Let $h\colon [0,1] \to X$ be defined by $h(t) = H(0,t)$, so that h keeps track of the basepoint change along H. Then $h(0) = H(0,0) = f(0)$ and $h(1) = H(0,1) = g(0)$, so $h \in \pi_1(X, x_0)$. We claim that $hgh^{-1} \simeq f$. Write

$$f \simeq \begin{cases} h(3t) & \text{if } 0 \le t \le 0 \\ H(t,0) & \text{if } 0 \le t \le 1 \\ h^{-1}(3t-2) & \text{if } 1 \le t \le 1 \end{cases},$$

   and

$$hgh^{-1} \simeq \begin{cases} h(3t) & \text{if } 0 \le t \le \frac{1}{3} \\ H(3(t-\frac{1}{3}),1) & \text{if } \frac{1}{3} \le t \le \frac{2}{3} \\ h^{-1}(3t-2) & \text{if } \frac{2}{3} \le t \le 1 \end{cases}.$$

   This observation suggests using the following homotopy $H'(t,s)\colon [0,1]^2 \to X$ from f to hgh^{-1}:

$$H'(t,s) = \begin{cases} h(3t) & \text{if } 0 \le t \le \frac{s}{3} \\ H((2s+1)(t-\frac{s}{3}),s) & \text{if } \frac{s}{3} \le t \le 1-\frac{s}{3} \\ h^{-1}(3t-2) & \text{if } 1-\frac{s}{3} \le t \le 1 \end{cases}.$$

   Then $H'(t,0) = f(t)$, and $H'(t,1) = hgh^{-1}$, and $H'(0,s) = H'(1,s) = h(0) = x_0$, so f and g come from the same conjugacy class of $\pi_1(X, x_0)$, and hence Φ is injective. □

16. **Exercise.** Show that there are no retractions $r\colon X \to A$ in the following cases:

(a) $X = \mathbf{R}^3$ with A any subspace homeomorphic to S^1.

(b) $X = \mathrm{S}^1 \times D^2$ with A its boundary torus $\mathrm{S}^1 \times \mathrm{S}^1$.

(c) $X = \mathrm{S}^1 \times D^2$ with A the circle shown in the figure (refer to Hatcher p.39).

(d) $X = D^2 \vee D^2$ with A its boundary $\mathrm{S}^1 \vee \mathrm{S}^1$.

(e) X a disk with two points on its boundary identified and A its boundary $\mathrm{S}^1 \vee \mathrm{S}^1$.

(f) X the Möbius band and A its boundary circle.

Solution. If there is a retraction $r\colon X \to A$ and $i\colon A \to X$ is inclusion, then ri is the identity on A, and the induced homomorphism $r_* i_*$ is the identity homomorphism on $\pi_1(A)$, so i_* is injective.

(a) Since $A \cong \mathrm{S}^1$, $\pi_1(A) \cong \mathbf{Z}$. Also, $\pi_1(\mathbf{R}^3) \cong 0$, and there is no injection $\mathbf{Z} \to 0$, so A cannot be a retraction of $\mathbf{R}^3$.

(b) Since $\pi_1(\mathrm{S}^1) \cong \mathbf{Z}$ and $\pi_1(D^2) \cong 0$, we get $\pi_1(\mathrm{S}^1 \times D^2) \cong \mathbf{Z}$ and $\pi_1(\mathrm{S}^1 \times \mathrm{S}^1) \cong \mathbf{Z} \times \mathbf{Z}$ because both S^1 and D^2 are path-connected. For any homomorphism $f\colon \mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$, we have $f((1,0)) = n$ and $f((0,1)) = m$ for some integers m and n. But then $f((m,0)) = nm$ and $f((0,n)) = nm$, so f cannot be injective. Thus, there is no retraction of $\mathrm{S}^1 \times D^2$ to $\mathrm{S}^1 \times \mathrm{S}^1$.

(c) As above, $\pi_1(\mathrm{S}^1 \times D^2) \cong \mathbf{Z}$, and since A is homeomorphic to S^1, $\pi_1(A) \cong \mathbf{Z}$. Let x_0 be some point of A. The homomorphism $i_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \to X$ is given by mapping $h\colon [0,1] \to A$ to the composition $[0,1] \xrightarrow{h} A \xrightarrow{i_*} X$. However, if h is the generator of $\pi_1(A, x_0)$ that loops around A once, then $i_*(h)$ is nullhomotopic, so i_* is not injective. This gives that no retraction of X onto A can exist.

(d) Since D^2 is contractible, each copy of D^2 can be contracted to the identified point in $D^2 \vee D^2$, and thus $D^2 \vee D^2$ has trivial fundamental group. However, the fundamental group of $\mathrm{S}^1 \vee \mathrm{S}^1$ is F_2, the free group on 2 generators, by the van Kampen theorem. Since there is no injection of F_2 into the trivial group, there cannot be a retraction of $D^2 \vee D^2$ onto its boundary.

(e) Up to homeomorphism, we may assume that the disk is the unit disk in $\mathbf{R}^2$ and that the two points that are identified are $(1,0)$ and $(-1,0)$. There is a homotopy from X to the circle $[-1,1]$ on the x-axis via the map $h_t((x,y)) = (x, (1-t)y)$, so $\pi_1(X) \cong \mathbf{Z}$. However, $\pi_1(\mathrm{S}^1 \vee \mathrm{S}^1) \cong F_2$. If a and b are the generators of F_2, and $f\colon F_2 \to \mathbf{Z}$ is a homomorphism, then $f(a) = n$ and $f(b) = m$ for some integers n and m. Then $f(a^m) = mn$ and $f(b^n) = mn$, but $a^m \neq b^n$, so there is no injection $F_2 \to \mathbf{Z}$, and thus no retract of X onto its boundary.

(f) Let X be the Möbius band and A its boundary. The inclusion $i\colon A \to X$ induces a homomorphism $i_*\colon \pi_1(A) \to \pi_1(X)$. Both groups are $\mathbf{Z}$, and $i_*(x) = 2x$ because looping around the boundary of the Möbius band is the same as looping twice around the Möbius band itself. This can be seen by letting A be the horizontal sides of a square whose vertical sides are identified with opposite orientation. If a retraction $r\colon X \to A$ exists, then ri is the identity on A, so by functoriality, $r_* i_*$ is the identity homomorphism on $\pi_1(A)$. If this were the case, then $i_*(1) = 2$, and $r_*(2) = 1$, which implies $r_*(1) + r_*(1) = 1$, but $r_*(1)$ cannot have an integer value. Thus there is no retraction of the Möbius band to its boundary. □

1.2 Van Kampen's Theorem

17. **Exercise.** Show that $\pi_1(\mathbf{R}^2 \setminus \mathbf{Q}^2)$ is uncountable.

Solution. To see that $\mathbf{R}^2 \setminus \mathbf{Q}^2$ is path-connected, choose two points (a, b) and (c, d). Either a or b must be irrational. Same with c and d. If a and d are both irrational, there is straight line path from (a, b) to (a, d), and then another straight line path from (a, d) to (c, d). If instead c is irrational, there is a straight line path from (c, d) to (c, d') where d' is some irrational number, and this is the previous case. The other cases are similar, so we can compute $\pi_1(\mathbf{R}^2 \setminus \mathbf{Q}^2)$ for any base point we like. For each irrational number α, let B_α be the union of $\{(x, \sqrt{2}) : -\alpha \le x \le \alpha\}$, $\{(x, -\sqrt{2}) : -\alpha \le x \le \alpha\}$, $\{(\alpha, y) : -\sqrt{2} \le y \le \sqrt{2}\}$, and $\{(-\alpha, y) : -\sqrt{2} \le y \le \sqrt{2}\}$. Neither of these sets contains a point of $\mathbf{Q}^2$, so we think of it as a box in $\mathbf{R}^2 \setminus \mathbf{Q}^2$. Let h_α be a loop based at $(0, \sqrt{2})$ that goes along B_α counterclockwise. If $\alpha < \beta$ are two irrationals, we claim that h_α and h_β are not homotopic to one another. The interior of the loop $h_\alpha h_\beta^{-1}$ can be thought of as the space outside of h_α and inside h_β. To be more precise, we mean the set of points $(x, y) \in \mathbf{R}^2 \setminus \mathbf{Q}^2$ such that $|y| < \sqrt{2}$ and $\alpha < |x| < \beta$. By the denseness of $\mathbf{Q}$ in $\mathbf{R}$, there is a rational number q in between α and β. Consider the inclusion $\mathbf{R}^2 \setminus \mathbf{Q}^2 \to \mathbf{R}^2 \setminus \{(0, q)\}$. This induces a homomorphism $\varphi\colon \pi_1(\mathbf{R}^2 \setminus \mathbf{Q}^2) \to \pi_1(\mathbf{R}^2 \setminus \{(0, q)\})$. Then $\varphi(h_\alpha h_\beta^{-1})$ is the same path in $\mathbf{R}^2 \setminus \{(0, q)\}$. This space is homotopic to S^1 and under such a homotopy from $\mathbf{R}^2 \setminus \{(0, q)\}$ to S^1, $\varphi(h_\alpha h_\beta^{-1})$ becomes a nontrivial loop around S^1, so is not nullhomotopic. Thus $h_\alpha h_\beta^{-1}$ cannot be nullhomotopic because φ is a homomorphism, so h_α and h_β are different elements in $\pi_1(\mathbf{R}^2 \setminus \mathbf{Q}^2)$. We have exhibited an injection of the irrationals into $\pi_1(\mathbf{R}^2 \setminus \mathbf{Q}^2)$, and since the set of irrational numbers is uncountable, we have the desired result. □

1.3 Covering Spaces

1. **Exercise.** For a covering space $p\colon \widetilde{X} \to X$ and a subspace $A \subset X$, let $\widetilde{A} = p^{-1}(A)$. Show that the the restriction $p\colon \widetilde{A} \to A$ is a covering space.

   Solution. For each point $x \in A$, there is a neighborhood U in X such that $p^{-1}(X)$ is the disjoint union of open sets U_i in $\widetilde{X}$ each of which gets mapped homeomorphically to U. Also, $U \cap A$ is an open set, and the $U_i \cap \widetilde{A}$ form a disjoint union of $p^{-1}(U \cap A)$. Each $U_i \cap \widetilde{A}$ is mapped homeomorphically to $U \cap A$, so $\widetilde{A}$ is a covering space of A. □

2. **Exercise.** Show that if $p_1\colon \widetilde{X}_1 \to X_1$ and $p_2\colon \widetilde{X}_2 \to X_2$ are covering spaces, so is their product $p_1 \times p_2\colon \widetilde{X}_1 \times \widetilde{X}_2 \to X_1 \times X_2$.

   Solution. Choose $(x_1, x_2) \in X_1 \times X_2$. Then there is a neighborhood U_i of x_i in X_i such that $p_i^{-1}(U_i)$ is a disjoint union of open sets $V_{i,\alpha}$ in $\widetilde{X}_i$ which map homeomorphically to U_i. So $U_1 \times U_2$ is an open set of $X_1 \times X_2$ such that $(p_1 \times p_2)^{-1}(U_1 \times U_2)$ is a disjoint union of products $V_{1,\alpha} \times V_{2,\beta}$ each of which maps homeomorphically to $U_1 \times U_2$. □

14. **Exercise.** Find all the connected covering spaces of $\mathbf{RP}^2 \vee \mathbf{RP}^2$.

    Solution. Let $X = \mathbf{RP}^2 \vee \mathbf{RP}^2$, and let X_1 and X_2 denote the first and second copy of $\mathbf{RP}^2$. Since $\pi_1(\mathbf{RP}^2) = \mathbf{Z}/2$ (this is done in example 1.43 of Hatcher), using the van Kampen theorem, $\pi_1(X) = \mathbf{Z}/2 * \mathbf{Z}/2$. Let a and b be the generators of $\mathbf{Z}/2 * \mathbf{Z}/2$. To understand the connected covering spaces of X, we classify the proper subgroups of $\mathbf{Z}/2 * \mathbf{Z}/2$. We describe them in terms of their generators. The first observation is that every element of $\mathbf{Z}/2 * \mathbf{Z}/2$ is a word of alternating a and b. The words that start and end with the same letter are precisely the set of elements of order 2. The other words are of the form $(ab)^n$ and $(ba)^n$, and these two are inverse to one another, so without loss of generality, if $(ba)^n$ is in a generating set, it can be replaced by $(ab)^n$. For every $n \ge 0$, there is a cyclic subgroup generated by $(ab)^n$. In particular the subgroup generated by ab is cyclic, and hence all of its subgroups are cyclic, so any set of generators $\{(ab)^{n_1}, (ab)^{n_2}, \dots\}$ can be replaced with a single generator $(ab)^n$ for some n. The other subgroups have generating sets $\{(ab)^n, g\}$ where g is an element of order 2 and $n \geq 0$. Note that if g and g' are both of order 2, then $gg' = (ab)^m$ for some m, so a set $\{(ab)^n, g, g'\}$ generates the same subgroup as $\{(ab)^k, g\}$ for some k, and the same applies for infinitely many elements of order 2 (similar argument as for the case of generators $\{(ab)^{n_1}, (ab)^{n_2}, \dots\}$, so we have described all of the subgroups of $\mathbf{Z}/2 * \mathbf{Z}/2$.

The universal covering space $\widetilde{X}$ of X is the infinite chain of S^2 shown in Figure 4. We number them with $\mathbf{Z}$, and map the S^2 with odd numbering to X_1 and the others to X_2 via the canonical map $\mathrm{S}^2 \to \mathbf{RP}^2$. In each of the following cases, we will use this same map (we define the numbering in Figure 4). The covering space associated to the subgroup generated by $(ab)^n$ for $n > 0$ is a chain of S^2 of length $2n$. To get covering spaces associated to the subgroups of order 2, we can take one copy of $\mathbf{RP}^2$ and attach an infinite chain of S^2 to one end. How we number the S^2 and where we map the $\mathbf{RP}^2$ gives rise to different subgroups of order 2. Also, we can choose different base points. Of course, X itself corresponds to $\mathbf{Z}/2 * \mathbf{Z}/2$. Finally, for the group with generators $\{(ab)^n, g\}$ with $n > 0$ and g is an element of order 2, we can take a copy of $\mathbf{RP}^2$, attach a chain of S^2 of length $2(n-1)$, and attach to the end another copy of $\mathbf{RP}^2$. The covering map sends the first $\mathbf{RP}^2$ to X_1 and the second one to X_2. If we number the S^2 in the chain, then the odd ones go to X_2 and the even ones to X_1 via the antipode identification. Depending on which base point we choose, we can get the subgroups for various g. □

26. **Exercise.** For a covering space $p\colon \widetilde{X} \to X$ with X connected, locally path-connected, and semilocally simply-connected, show:

(a) The components of $\widetilde{X}$ are in one-to-one correspondence with the orbits of the action of $\pi_1(X, x_0)$ on the fiber $p^{-1}(x_0)$.

(b) Under the Galois correspondence between connected covering spaces of X and subgroups of $\pi_1(X, x_0)$, the subgroup corresponding to the component of $\widetilde{X}$ containing a given lift $\widetilde{x}_0$ of x_0 is the stabilizer of $\widetilde{x}_0$.

Solution.

(a) Choose $z_0, z_1 \in p^{-1}(x_0)$. If z_0 and z_1 are in different components of $\widetilde{X}$, $\pi_1(X, x_0)$ cannot map one to the other because there is no path connecting them. So we need to show that $\pi_1(X, x_0)$ acts transitively on each of the components of $\widetilde{X}$ to get the bijection. Since X is assumed locally path-connected, $\widetilde{X}$ is locally path-connected. Thus, the notions of connected components and path-connected components are the same. If z_0 and z_1 are in the same component, let γ be a path joining them. Then $p\gamma$ is an element of $\pi_1(X, x_0)$ whose action on $p^{-1}(x_0)$ maps z_1 to z_0 (by Hatcher's definition), and this gives the transitivity. Then the set of elements in $p^{-1}(x_0)$ in a given component form an orbit, and this gives the desired bijection.

(b) Choose a given lift $\widetilde{x}_0$ of x_0 in some component X' of $\widetilde{X}$. Under the Galois correspondence, the subgroup of $\pi_1(X, x_0)$ associated to X' is the image of $G = \pi_1(X', \widetilde{x}_0)$ in the inclusion $p_*\colon G \to \pi_1(X, x_0)$. Any loop $\gamma \in p_*G$ then lifts back to a loop in X' by the unique lifting property, so γ sends $\widetilde{x}_0$ to itself, and is an element of the stabilizer of $\widetilde{x}_0$. Conversely, if $\beta \in \pi_1(X, x_0)$ is in the stabilizer of $\widetilde{x}_0$, then the lift $\bar{\beta}$ of β is a loop from $\widetilde{x}_0$ to itself, so $\bar{\beta} \in G$, which means $\beta \in p_*G$. This gives that p_*G is the stabilizer of $\widetilde{x}_0$. □

1.A. Graphs and Free Groups

3. **Exercise.** For a finite graph X define the Euler characteristic $\chi(X)$ to be the number of vertices

minus the number of edges. Show that $\chi(X) = 1$ if X is a tree, and that the rank (number of elements in a basis) of $\pi_1(X)$ is $1 - \chi(X)$ if X is connected.

Solution. If X is a tree, then by Hatcher's definition, it is contractible to a point, so must be connected. Furthermore, for any two vertices v and w, there is a unique path from v to w. If not, going along one and then backwards along the other gives a loop that is not nullhomotopic, which contradicts the contractibility. Suppose X has n vertices. We claim that X has $n-1$ edges. If $n = 1$, this is clear. For a graph on n vertices, remove any edge e. Then the remaining space has two connected components. If not, the endpoints of e have another path connecting them, which is a contradiction. The connected components have k and $n-k$ vertices, and are trees. This follows because X can be contracted to any vertex, so the connected components are also contractible. By induction, the connected components have $k-1$ and $n-k-1$ edges, so X has $(k-1)+(n-k-1)+1 = n-1$ edges, and $\chi(X) = 1$.

Let T be a maximal tree in X. The existence of maximal trees is given by Proposition 1A.1 in Hatcher. For each edge e_α in $X \setminus T$, we can choose a small neighborhood U_α of $T \cup e_\alpha$ in X that deformation retracts onto $T \cup e_\alpha$. Then $\{U_\alpha\}$ is a covering of X, and the intersection of any of them contains a small neighborhood of T, so is path-connected. Since T is contractible to a point, $\pi_1(T) = 0$, so the van Kampen theorem gives $\pi_1(X) \cong *_\alpha \pi_1(U_\alpha)$. Each U_α deformation retracts to $T \cup e_\alpha$, which is homotopy equivalent to S^1 because it contains exactly one cycle (this follows from the uniqueness of paths in T), so $\pi_1(U_\alpha) \cong \mathbf{Z}$. Thus $\pi_1(X)$ is a free group whose rank is the number of edges of X minus the number of edges of T. If e is the number of edges of X and v is the number of vertices, then the rank of $\pi_1(X)$ is $e - (v-1)$ since T has $v-1$ edges, and this is equal to $1 - \chi(X)$. □

6. **Exercise.** Let F be the free group on two generators and let F' be its commutator subgroup. Find a set of free generators for F' by considering the covering space of the graph $\mathrm{S}^1 \vee \mathrm{S}^1$ corresponding to F'.

Solution. Let a and b denote the generators of F. Construct a graph $\widetilde{X}$ whose vertices are the integer points $\mathbf{Z}^2$ in the plane, with an edge in between (x, y) and (x', y') if and only if $|x - x'| + |y - y'| = 1$. Let the base point be $\widetilde{x}_0 = (0, 0)$. Any edge either connects (x, y) to $(x + 1, y)$, or (x, y) to $(x, y + 1)$ for some values of x and y. In both cases, orient the edge away from (x, y). In the first case, label this edge with an a, and with the second, label it with a b. We show part of this graph in Figure 5. Then this orientation is well-defined, and every vertex has exactly one a edge coming in, one a edge coming out, and the same with b, so $(\widetilde{X}, \widetilde{x}_0)$ is a covering space of (X, x_0) where $X = \mathrm{S}^1 \vee \mathrm{S}^1$ and x_0. The covering map p maps each a to a loop going around one of the copies of S^1 and each b to a loop going around the other copy. For $m \in \mathbf{Z}$ with $m > 0$, define a_m to be the edges of the form $\{(x, m), (x + 1, m)\}$ where $0 \le x < m$ and define b_m to be the edges of the form $\{(m, y), (m, y + 1)\}$ where $0 \le y < m$. Also, define a_{-m} to be the edges making up the reflection of a_m across the y-axis, and b_{-m} to be the edges making up the reflection of b_m across the x-axis. Then for any $m, n \in \mathbf{Z} \setminus \{0\}$, define $X_{m,n}$ to be the union of a_n, b_m, and the edges on the x-axis and y-axis. We illustrate this in Figure 5. It is clear that the union of the $X_{m,n}$ is $\widetilde{X}$, and their intersection is the union of the x-axis and the y-axis, whose fundamental group is trivial. Also, each triple intersection is path-connected since it is the union of the x-axis and the y-axis and possibly also some sets a_m and b_n for some numbers m and n. Note that $\pi_1(X_{m,n}, \widetilde{x}_0)$ is a free group generated by the loop that goes from $(0, 0)$ to $(0, n)$ to (m, n) to $(m, 0)$ and then back to $(0, 0)$. Also, $(X_{m,n}, \widetilde{x}_0)$ is a covering space for X, and the subgroup it maps to in $\pi_1(X, x_0)$ is the one generated by $[a^n, b^m]$. By van Kampen's theorem, the image of $\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $\pi_1(X, x_0)$ is a free group generated by the elements $\{[a^n, b^m] : m, n \in \mathbf{Z} \setminus \{0\}\}$. The commutator subgroup G of F contains this set, and is generated by it, so we conclude that it is freely generated by this set. □

2 Homology

2.1. Simplicial and Singular Homology

8. **Exercise.** Construct a 3-dimensional Δ-complex X from n tetrahedra $T_1, \ldots, T_n$ by the following two steps. First arrange the tetrahedra in a cyclic pattern as in the figure (refer to Hatcher, p.131), so that each T_i shares a common vertical face with its two neighbors T_{i-1} and T_{i+1}, the subscripts being taken mod n. Then identify the bottom face of T_i with the top face of T_{i+1} for each i. Show the simplicial homology groups of X in dimensions 0, 1, 2, 3 are $\mathbf{Z}$, $\mathbf{Z}_n$, 0, $\mathbf{Z}$, respectively.

Solution. All of the outer vertices are identified with one another, and the two inner vertices are also identified, so X has 2 0-cells. Label the outer vertex v_0 and the inner vertex v_1. Also, the n 3-cells each have 4 2-dimensional faces, but they are identified in pairs, so there are $2n$ 2-cells. Each of the edges connecting v_0 to itself are identified, and there is only one edge connecting v_1 to itself. So each tetrahedron has 4 remaining edges. Each one is identified with an edge of its neighbor, and then further identified with another edge by identifying the bottom face of T_i with the top face of T_{i+1}, so in total there are only n 1-cells that connect v_0 and v_1. Thus, we compute the homology of the complex

$$0 \longrightarrow \mathbf{Z}^n \xrightarrow{\partial_3} \mathbf{Z}^{2n} \xrightarrow{\partial_2} \mathbf{Z}^{n+2} \xrightarrow{\partial_1} \mathbf{Z}^2 \longrightarrow 0\,.$$

We order the faces of X based on the orientation of the edges in the figure in Hatcher. Each 1-cell either connects v_0 to itself, v_1 to itself, or connects v_0 to v_1. In the first two cases, ∂_1 maps these 1-cells to 0, and in the last case ∂_1 maps them to $v_0 - v_1$, so $\partial_1(\mathbf{Z}^{n+2}) \cong \mathbf{Z}$, which means $\mathrm{H}_0(X) = \mathbf{Z}^2/\mathbf{Z} \cong \mathbf{Z}$.

Label the bottom face of T_i as f_i and label its face on the right side (using counterclockwise orientation in the figure in Hatcher) f_{n+i}. Also, label the outer edge e and the edge connecting v_1 to itself e_{n+1}. Label the bottom edge of f_{n+i} with e_i. For $1 \le i \le n$, we have $\partial_2(f_i) = -e_i + e_{i-1} - e$ and $\partial_2(f_{n+i}) = -e_{n+1} + e_{i-1} - e_i$, where e_0 means e_n. If we order the edges $e, e_1, \ldots, e_n, e_{n+1}$, then the image of ∂_2 is the subgroup generated by the row vectors of the following $2n \times (n+2)$ matrix

$$\begin{bmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ -1 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & \cdots & 0 & 0 & -1 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & -1 \end{bmatrix}. \tag{1}$$

For $1 \leq i \leq n$, we subtract row i from row $n+i$ and see that the resulting last n rows are the same, so we can reduce to the following $(n+1) \times (n+2)$ matrix

$$\begin{bmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ -1 & 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$

Now for $1 \leq i \leq n$, add the first $i-1$ rows to the ith row to get

$$\begin{bmatrix} -1 & -1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & \cdots & 0 & 1 & 0 \\ -3 & 0 & 0 & -1 & \cdots & 0 & 1 & 0 \\ & & & & \vdots & & & \\ -n & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \end{bmatrix}.$$

Note that e and e_{n+1} are the only edges mapped to 0 by ∂_1 and every other edge is mapped to $v_0 - v_1$, so $\ker \partial_1$ is generated by e, e_{n+1}, and all differences $e_i - e_j$ where $1 \leq i < j \leq n$. These differences can be generated by just $e_n - e_i$ for $1 \leq i \leq n-1$, so we have $\ker \partial_1 = \langle e, e_{n+1}, e_n - e_i \rangle = \langle e, e - e_{n+1}, -ie + e_n - e_i \rangle$ where the second equality is a result of adding e some number of times to each generator. From this, it is easy to see that $\mathrm{H}_1(X) = \ker \partial_1 / \operatorname{image} \partial_2 \cong \mathbf{Z}/n$.

For any $a_1 f_1 + \cdots + a_{2n} f_{2n} \in \ker \partial_2$, reading off from (1), we see that $a_1 + \cdots + a_n = a_{n+1} + \cdots + a_{2n} = 0$ and that for $1 < i \leq n$, we have $a_i + a_{n+i} = a_{i-1} + a_{n+i-1}$ and $a_1 + a_{n+1} = a_n + a_{2n}$. This implies that $a_1 + a_{n+1} = a_2 + a_{n+2} = \cdots = a_n + a_{2n}$; the sum of all of these terms is 0, so $a_i = -a_{n+i}$ for $1 \leq i \leq n$. We claim that $a_1 f_1 + \cdots + a_{2n} f_{2n} \in \operatorname{image} \partial_3$. Since

$$\begin{aligned} \partial_3(b_1 T_1 + \cdots + b_n T_n) &= b_1(-f_{n+1} + f_{2n} - f_n + f_1) + \cdots + b_n(-f_{2n} + f_{2n-1} - f_{n-1} + f_n) \\ &= (b_1 - b_2) f_1 + (b_2 - b_3) f_2 + \cdots + (b_n - b_1) f_n \\ &\quad + (b_2 - b_1) f_{n+1} + (b_3 - b_2) f_{n+2} + \cdots + (b_1 - b_n) f_{2n}, \end{aligned}$$

it is enough to find $b_1, \ldots, b_n$ such that $b_i - b_{i+1} = a_i$ where b_{n+1} means b_1. To do this, we can pick any b_1, and the other b_i are determined. The only thing to check is that $b_n - b_1 = a_n$, but this follows because

$$b_n - b_1 = -((b_1 - b_2) + (b_2 - b_3) + \cdots + (b_{n-1} - b_n)) = -(a_1 + \cdots + a_{n-1}) = a_n.$$

This gives $\operatorname{image} \partial_3 = \ker \partial_2$, so $\mathrm{H}_2(X) = 0$.

For each $1 < i \leq n$, $\partial_3(T_i) = -f_{n+i} + f_{n+i-1} - f_{i-1} + f_i$, and $\partial_3(T_1) = -f_{n+1} + f_{2n} - f_n + f_1$. Any 2-cell f_j appears in two neighboring 3-cells, say T_i and T_{i+1}. The coefficient of f_j in $\partial_3(T_i)$ and $\partial_3(T_{i+1})$ appear with opposite sign. Thus, if $a_1 T_1 + \cdots + a_n T_n \in \ker \partial_3$, then $a_1 = a_2 = \cdots = a_n$. So $\ker \partial_3 \cong \mathbf{Z}$, which gives $\mathrm{H}_3(X) \cong \mathbf{Z}/0 = \mathbf{Z}$. □

12. **Exercise.** Show that chain homotopy of chain maps is an equivalence relation.

Solution. Let (C, ∂) and (C', ∂') be two chain complexes. If $f, g \colon C \to C'$ are two chain maps, write $f \sim g$ if f is chain homotopic to g, i.e., there are maps $s \colon C_n \to C'_{n+1}$ such that

$f - g = s\partial + \partial' s$. By choosing $s = 0$, we see that $f \sim f$. If $f \sim g$ via the map s, then $g \sim f$ via the map $-s$:

$$g - f = -(f - g) = -(s\partial + \partial' s) = (-s)\partial + \partial'(-s).$$

Finally, if $f \sim g$ via s and $g \sim h$ via t, then

$$f - h = (f - g) + (g - h) = (s\partial + \partial' s) + (t\partial + \partial' t) = (s + t)\partial + \partial'(s + t),$$

so $f \sim h$ via $s + t$. Thus, chain homotopy is an equivalence relation. □

18. **Exercise.** Show that for the subspace $\mathbf{Q} \subset \mathbf{R}$, the relative homology group $\mathrm{H}_1(\mathbf{R}, \mathbf{Q})$ is free Abelian and find a basis.

Solution. The long exact sequence on homology gives

$$\cdots \longrightarrow \mathrm{H}_1(\mathbf{R}) \longrightarrow \mathrm{H}_1(\mathbf{R}, \mathbf{Q}) \longrightarrow \mathrm{H}_0(\mathbf{Q}) \xrightarrow{f} \mathrm{H}_0(\mathbf{R}) \longrightarrow \cdots.$$

Since $\mathbf{R}$ is contractible, $\mathrm{H}_1(\mathbf{R}) = 0$, so $\mathrm{H}_1(\mathbf{R}, \mathbf{Q}) = \ker f$. For a space X, $\mathrm{H}_0(X)$ is a direct sum of $\mathbf{Z}$ with one copy for each connected component. So $f\colon \bigoplus_{x\in\mathbf{Q}} \mathbf{Z} \to \mathbf{Z}$ is induced by the inclusion $\mathbf{Q} \hookrightarrow \mathbf{R}$ and the images of each generator of $\mathrm{H}_0(\mathbf{Q})$ are all homologous to a generator of $\mathrm{H}_0(\mathbf{R})$. This implies that $\ker f$ is the set of elements $a_1 e_{p_1} + \cdots + a_r e_{p_r}$ such that $a_1 + \cdots + a_r = 0$, and this set is generated by elements of the form $e_p - e_q$ where e_p denotes the identity element in the copy of $\mathbf{Z}$ corresponding to $p \in \mathbf{Q}$ (reason: $a_1 e_{p_1} + \cdots + a_r e_{p_r} = a_1(e_{p_1} - e_{p_2}) + (a_1 + a_2)(e_{p_2} - e_{p_3}) + \cdots + (a_1 + a_2 + \cdots + a_{r-1})(e_{p_{r-1}} - e_{p_r})$ if $a_1 + \cdots + a_r = 0$). We claim that a basis for the kernel is $B := \{e_0 - e_p : p \in \mathbf{Q} \setminus \{0\}\}$. It is clear that B is linearly independent: if there is a relation $a_1(e_0 - e_{p_1}) + \cdots + a_r(e_0 - e_{p_r})$, then $a_1 e_0 + \cdots + a_r e_0 = 0$ since there are no relations among e_0 and the e_{p_i}, so $a_i = 0$ for all i. To see B generates $\ker f$, pick any $e_p - e_q$. Then $(e_0 - e_q) - (e_0 - e_p) = e_p - e_q$, so any element generated by the $e_p - e_q$ can be generated by B. □

27. **Exercise.** Let $f\colon (X, A) \to (Y, B)$ be a map such that both $f\colon X \to Y$ and the restriction $f\colon A \to B$ are homotopy equivalences.

   (a) Show that $f_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$ is an isomorphism for all n.

   (b) For the case of the inclusion $f\colon (D^n, \mathrm{S}^{n-1}) \to (D^n, D^n \setminus \{0\})$, show that f is not a homotopy equivalence of pairs – there is no $g\colon (D^n, D^n \setminus \{0\}) \to (D^n, \mathrm{S}^{n-1})$ such that fg and gf are homotopic to the identity through maps of pairs.

Solution.

(a) By naturality of the long exact sequence of homology (p.127 of Hatcher), for all n, the diagram

$$\begin{array}{ccccccccc}
\mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \longrightarrow & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}(X) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
\mathrm{H}_n(B) & \longrightarrow & \mathrm{H}_n(Y) & \longrightarrow & \mathrm{H}_n(Y, B) & \longrightarrow & \mathrm{H}_{n-1}(B) & \longrightarrow & \mathrm{H}_{n-1}(Y)
\end{array}$$

commutes. Since $f\colon X \to Y$ and $f\colon A \to B$ are homotopy equivalences, the first two and last two vertical arrows in the above diagram are isomorphisms. Also, the top and bottom rows are exact, so by the five-lemma, the middle vertical arrow is also an isomorphism.

(b) Let $g\colon (D^n, D^n \setminus \{0\}) \to (D^n, S^{n-1})$ be a map of pairs. Since S^{n-1} is a closed set, $g^{-1}(S^{n-1})$ is closed. By assumption, $D^n \setminus \{0\} \subseteq g^{-1}(S^{n-1})$. Also, 0 is in the closure of $D^n \setminus \{0\}$, so $g(0) \in S^{n-1}$, and thus $D^n \setminus \{0\} \hookrightarrow D^n \to S^{n-1}$ is a factorization of g. We have an induced map on homology $H_n(D^n \setminus \{0\}) \to H_n(D^n) \to H_n(S^{n-1})$ whose composition is g_*. Since $H_n(D^n) = 0$ (here we assume $n > 0$, but if $n = 0$, then $D^0 \setminus \{0\} = \varnothing$, so f won't exist), we conclude that $g_* = 0$. Thus g does not induce an isomorphism on homology because $H_n(S^{n-1}) = \mathbf{Z}$, so g cannot be a homotopy equivalence of pairs. Since g was arbitrary, we see that f is also not a homotopy equivalence of pairs. □

2.2. Computations and Applications

2. **Exercise.** Given a map $f\colon S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ has a fixed point. Construct maps $\mathbf{RP}^{2n-1} \to \mathbf{RP}^{2n-1}$ without fixed points from linear transformations $\mathbf{R}^{2n} \to \mathbf{R}^{2n}$ without eigenvectors.

Solution. Suppose there is a map $\varphi\colon S^{2n} \to S^{2n}$ such that φ has no fixed points and $\varphi(x) \neq -x$ for all $x \in S^{2n}$. Since φ has no fixed points, the line segment $(1-t)\varphi(x) - tx$ never passes through the origin, so we can define a homotopy from φ to the antipodal map with $F\colon I \times S^{2n} \to S^{2n}$ by

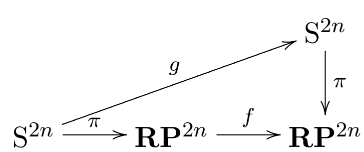
$$(t, x) \mapsto \frac{(1-t)\varphi(x) - tx}{|(1-t)\varphi(x) - tx|},$$

so $\deg \varphi = (-1)^{2n+1} = -1$. Similarly, the line segment $(1-t)\varphi(x) + tx$ never passes through the origin since $\varphi(x) \neq -x$ for all $x \in S^{2n}$, so we can define a homotopy from φ to the identity map with $F\colon I \times S^{2n} \to S^{2n}$ by

$$(t, x) \mapsto \frac{(1-t)\varphi(x) + tx}{|(1-t)\varphi(x) + tx|}.$$

Then $\deg \varphi = 1$, which is a contradiction. Thus, there must exist $x \in S^{2n}$ such that either $\varphi(x) = x$ or $\varphi(x) = -x$.

Now let $f\colon \mathbf{RP}^{2n} \to \mathbf{RP}^{2n}$ be any map. Composing it with the canonical map $\pi\colon S^{2n} \to \mathbf{RP}^{2n}$, we get $f'\colon S^{2n} \to \mathbf{RP}^{2n}$. Since S^{2n} is a covering space of $\mathbf{RP}^{2n}$, f' lifts (via the lifting criterion since S^{2n} has trivial fundamental group) to a map $g\colon S^{2n} \to S^{2n}$. In other words, the diagram



commutes. From above, there is a point $x \in S^{2n}$ such that either $g(x) = x$ or $g(x) = -x$. Then

$$f(\pi(x)) = \pi(g(x)) = \pi(\pm x) = \pi(x),$$

so $\pi(x)$ is a fixed point of f.

Let $T\colon \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ be the linear transformation given by $(x_1, x_2, \ldots, x_{2n}) \mapsto (-x_{2n}, x_1, x_2, \ldots, x_{2n-1})$. Then $T^{2n} = -I$, where I is the identity map on $\mathbf{R}^{2n}$, so $x^{2n} + 1$ divides the characteristic polynomial of T, and hence is the characteristic polynomial since it has degree $2n$. However, it has no real roots, so T has no real eigenvalues, and hence no eigenvectors. Notice that T acts on $S^{2n-1} \subseteq \mathbf{R}^{2n}$, and this action is a continuous map. Since T has no eigenvectors, we have $T(x) \neq x$ and $T(x) \neq -x$ for all $x \in S^{2n}$. Also, since $T(-x) = -T(x)$, T gives a well-defined map $\mathbf{RP}^{2n-1} \to \mathbf{RP}^{2n-1}$ which has no fixed points. □

8. **Exercise.** A polynomial $f(z)$ with complex coefficients, viewed as a map $\mathbf{C} \to \mathbf{C}$, can always be extended to a continuous map of one-point compactifications $\hat{f}\colon \mathrm{S}^2 \to \mathrm{S}^2$. Show that the degree of $\hat{f}$ equals the degree of f as a polynomial. Show also that the local degree at $\hat{f}$ at a root of f is the multiplicity of the root.

   Solution. Let $z_1, \ldots, z_r$ be the distinct roots of $f(z)$ with multiplicities $m_1, \ldots, m_r$. We can find disjoint neighborhoods $U_1, \ldots, U_r$ in S^2 for each z_i. Each U_i is mapped into some neighborhood V_i of 0. Consider the induced map on relative homology $\hat{f}_*\colon \mathrm{H}_2(U_i, U_i \setminus \{z_i\}) \to \mathrm{H}_2(V_i, V_i \setminus \{0\})$. Both groups are isomorphic to $\mathbf{Z}$, so this map is given by multiplication by some number, which is the local degree of $\hat{f}$ at z_i (see Hatcher p.136). The restriction of $\hat{f}$ to any small neighborhood of z_i will be a m_i-to-1 mapping onto some open neighborhood of 0 contained in its image. This implies that the local degree is m_i since a generator for $\mathrm{H}_2(U_i, U_i \setminus \{z_i\})$ is mapped to m_i times a generator of $\mathrm{H}_2(V_i, V_i \setminus \{0\})$.

   Since the local degree of $\hat{f}$ at z_i is m_i, we see that $\deg \hat{f} = \sum_i \deg \hat{f}|z_i = \sum_i m_i = \deg f$, where the first equality is by Proposition 2.30 of Hatcher and the last equality follows from the fundamental theorem of algebra. □

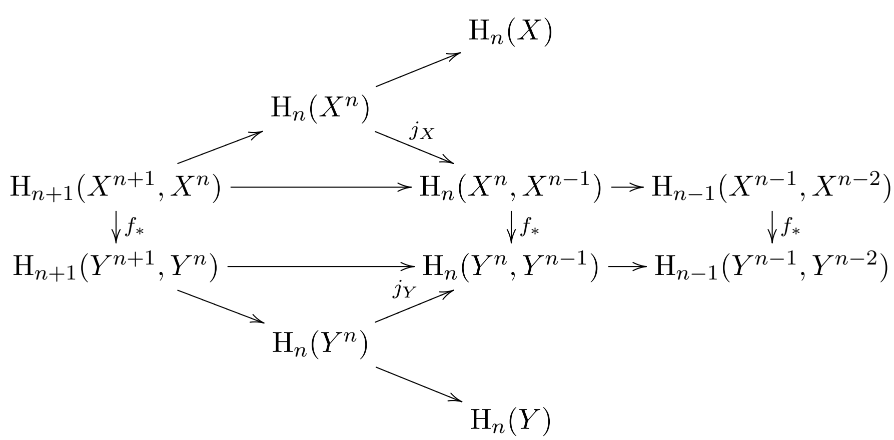
17. **Exercise.** Show the isomorphism between cellular and singular homology is natural in the following sense: A map $f\colon X \to Y$ that is *cellular* – satisfying $f(X^n) \subset Y^n$ for all n – induces a chain map f_* between the cellular chain complexes of X and Y, and the map $f_*\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)$ induced by this chain map corresponds to $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ under the isomorphism $\mathrm{H}_n^{\mathrm{CW}} \cong \mathrm{H}_n$.

    Solution. Since $f\colon X \to Y$ is a cellular map, for all $n \geq 0$, the restriction of f to the n-skeleton of X gives a map of pairs $(X^n, X^{n-1}) \to (Y^n, Y^{n-1})$, which induces a map on relative homology $f_*\colon \mathrm{H}_n(X^n, X^{n-1}) \to \mathrm{H}_n(Y^n, Y^{n-1})$. But cellular chain groups are defined as these relative homology groups, so f induces a chain map between the cellular chain complexs of X and Y.

    Also, f_* induces a map on homology $f_*^{\mathrm{CW}}\colon \mathrm{H}_n^{\mathrm{CW}}(X) \to \mathrm{H}_n^{\mathrm{CW}}(Y)$. By abuse of notation, $f\colon X \to Y$ induces a map on homology $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$. Let $i_X\colon \mathrm{H}_n(X) \to \mathrm{H}_n^{\mathrm{CW}}(X)$ and $i_Y\colon \mathrm{H}_n(Y) \to \mathrm{H}_n^{\mathrm{CW}}(Y)$ be the isomorphism between cellular and singular homology given by Theorem 2.35 of Hatcher. We wish to show that the diagram

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y) \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ \mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{f_*^{\mathrm{CW}}} & \mathrm{H}_n^{\mathrm{CW}}(Y) \end{array} \tag{2}$$

    commutes. In fact, the diagram without f_* comes from the commutative diagram



which is an augmentation of the one found on p.139 of Hatcher. Here the isomorphisms i_X and i_Y are induced by j_X and j_Y. If $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ fills in the above diagram, then we are done. But now this is just a question of filling in ? with f_* in the following diagram

$$\begin{array}{ccccc} \mathrm{H}_{n+1}(X^{n+1}, X^n) & \longrightarrow & \mathrm{H}_n(X^n) & \longrightarrow & \mathrm{H}_n(X) \\ \downarrow f_* & & & & \downarrow ? \\ \mathrm{H}_{n+1}(Y^{n+1}, Y^n) & \longrightarrow & \mathrm{H}_n(Y^n) & \longrightarrow & \mathrm{H}_n(Y) \end{array}$$

which is the long exact sequence on homology of good pairs. By the naturality of the long exact sequence on homology, we conclude that f_* does fill in ?, so (2) commutes. We conclude that the isomorphism between cellular homology and singular homology is natural. □

20. **Exercise.** For finite CW complexes X and Y, show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

    Solution. Given two finite CW complexes X and Y with some given CW structure, let a_n and b_n be the number of n-cells in X and Y, respectively. By the isomorphism of cellular homology and singular homology, we have $\chi(X) = \sum_n (-1)^n a_n$ and $\chi(Y) = \sum_n (-1)^n b_n$. The product $X \times Y$ has a CW structure whose cells are given by $e^m_\alpha \times e^n_\beta$ where e^m_α ranges over the cells of X and e^n_β ranges over the cells of Y (see Hatcher p.8). This gives $\chi(X \times Y) = \sum_n \sum_{i+j=n} (-1)^n a_i b_j$, and this is also the product $\chi(X)\chi(Y)$. □

21. **Exercise.** If a finite CW complex X is the union of subcomplexes A and B, show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.

    Solution. Now suppose X is a finite CW complex that is the union of two subcomplexes A and B. Let a_n, b_n, and c_n denote the number of n-cells in A, B, and $A \cap B$, respectively. By inclusion-exclusion, the number of n-cells in X is then $a_n + b_n - c_n$. So we have the following equalities:

$$\begin{aligned} \chi(X) &= \sum_n (-1)^n (a_n + b_n - c_n) \\ &= \sum_n (-1)^n a_n + \sum_n (-1)^n b_n - \sum_n (-1)^n c_n \\ &= \chi(A) + \chi(B) - \chi(A \cap B). \quad \square \end{aligned}$$

22. **Exercise.** For X a finite CW complex and $p\colon \widetilde{X} \to X$ an n-sheeted covering space, show that $\chi(\widetilde{X}) = n\chi(X)$.

    Solution. Now suppose that X is a finite CW complex and $p\colon \widetilde{X} \to X$ is an n-sheeted covering space. Then $\widetilde{X}$ has a CW complex structure where the i-cells are the lifts of i-cells of X. More specifically, every i-cell σ is equipped with a characteristic map $f_\sigma\colon D^i \to X$ which lifts to a unique map $\widetilde{f_\sigma}\colon D^i \to X$ once the image of any point is specified. Since p is n-sheeted, we can get n different lifts, so the number of i-cells of $\widetilde{X}$ is n times the number of i-cells of X. This gives the formula $\chi(\widetilde{X}) = n\chi(X)$, which follows directly from the alternating sum of number of cells. □

3 Cohomology

3.2. Cup Product

1. **Exercise.** Assuming as known the cup product structure on the torus $\mathrm{S}^1 \times \mathrm{S}^1$, compute the

cup product structure in $\mathrm{H}^*(M_g)$ for M_g the closed orientable surface of genus g by using the quotient map from M_g to a wedge sum of g tori.

Solution. From the universal coefficient theorem, we have the following exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_0(M_g;\mathbf{Z}),\mathbf{Z}) \longrightarrow \mathrm{H}^1(M_g;\mathbf{Z}) \longrightarrow \mathrm{Hom}(\mathrm{H}_1(M_g;\mathbf{Z}),\mathbf{Z}) \longrightarrow 0\ .$$

We know that $\mathrm{H}_1(M_g;\mathbf{Z}) \cong \mathbf{Z}^{2g}$, so $\mathrm{Hom}(\mathrm{H}_1(M_g;\mathbf{Z}),\mathbf{Z}) \cong \mathbf{Z}^{2g}$, and $\mathrm{H}_0(M_g;\mathbf{Z}) \cong \mathbf{Z}$, so $\mathrm{Ext}(\mathrm{H}_0(M_g;\mathbf{Z}),\mathbf{Z}) = 0$, which all implies that $\mathrm{H}^1(M_g;\mathbf{Z}) \cong \mathbf{Z}^{2g}$. Similarly, we have the short exact sequence

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_1(M_g;\mathbf{Z}),\mathbf{Z}) \longrightarrow \mathrm{H}^2(M_g;\mathbf{Z}) \longrightarrow \mathrm{Hom}(\mathrm{H}_2(M_g;\mathbf{Z}),\mathbf{Z}) \longrightarrow 0$$

which implies that $\mathrm{H}^2(M_g;\mathbf{Z}) \cong \mathbf{Z}$. By Example 3.15 of Hatcher, $\mathrm{H}^*(T^2;\mathbf{Z}) \cong \bigwedge \mathbf{Z}^2$ is the exterior algebra on two generators where $T^2 = \mathrm{S}^1 \times \mathrm{S}^1$. By Example 3.13 of Hatcher, there is an isomorphism of reduced cohomology rings

$$\widetilde{\mathrm{H}}^*(\bigvee_{i=1}^{g} T^2;\mathbf{Z}) \cong \prod_{i=1}^{g} \widetilde{\mathrm{H}}^*(T^2;\mathbf{Z}).$$

Now let $f\colon M_g \to \bigvee_{i=1}^g T^2$ be the quotient map illustrated in Exercise 3.2.1 of Hatcher. This induces a graded homomorphism of cohomology rings

$$f^*\colon \mathrm{H}^*(\bigvee_{i=1}^{g} T^2;\mathbf{Z}) \to \mathrm{H}^*(M_g;\mathbf{Z}).$$

Denote the $2g$ generators of $\mathrm{H}^1(M_g;\mathbf{Z})$ as $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ where α_i,β_i correspond to the generators of the ith coordinate in the product $\prod_{i=1}^g \mathrm{H}^1(T^2;\mathbf{Z})$, and the one generator of $\mathrm{H}^2(M_g;\mathbf{Z})$ as γ. Also denote the generators of $\mathrm{H}^1(\bigvee_{i=1}^g T^2;\mathbf{Z})$ as $a_1,\dots,a_g,b_1,\dots,b_g$ where a_i,b_i correspond to the ith component, and the generators of $\mathrm{H}^2(\bigvee_{i=1}^g T^2;\mathbf{Z})$ as $c_1,\dots,c_g$. With an appropriate choice of labels, we see that $f^*(a_i) = \alpha_i$, $f^*(b_i) = \beta_i$, and $f^*(c_i) = \gamma$ for all i.

From this direct product description, it is immediately clear that for $i \neq j$,

$$\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = \alpha_i \smile \beta_j = \beta_i \smile \alpha_j = 0.$$

From the fact that each $\mathrm{H}^*(T^2;\mathbf{Z})$ is the exterior algebra of $\mathbf{Z}^2$, we also verify that $\alpha_i \smile \alpha_i = \beta_i \smile \beta_i = 0$. Finally, since $a_i \smile b_i = -b_i \smile a_i = c_i$, we see that $f^*(a_i) \smile f^*(b_i) = -f^*(b_i) \smile f^*(a_i) = f^*(c_i)$, which implies that $\alpha_i \smile \beta_i = -\beta_i \smile \alpha_i = \gamma$. □

4. **Exercise.** Apply the Leftschetz fixed point theorem to show that every map $f\colon \mathbf{CP}^n \to \mathbf{CP}^n$ has a fixed point if n is even, using the fact that $f^*\colon \mathrm{H}^*(\mathbf{CP}^n;\mathbf{Z}) \to \mathrm{H}^*(\mathbf{CP}^n;\mathbf{Z})$ is a ring homomorphism. When n is odd show there is a fixed point unless $f^*(\alpha) = -\alpha$, for α a generator of $\mathrm{H}^2(\mathbf{CP}^n;\mathbf{Z})$.

Solution. The cohomology ring of $\mathbf{CP}^n$ is $\mathrm{H}^*(\mathbf{CP}^n;\mathbf{Z}) = \mathbf{Z}[x]/(x^{n+1})$ where x has degree 2 by Theorem 3.12 of Hatcher. So each cohomology group in even degree $\leq n$ has rank 1, and each cohomology group in odd degree is 0. By the naturality of the universal coefficient theorem (Hatcher p.196) and the discussion of trace in Hatcher p.181, the trace of $f^*\colon \mathrm{H}^i(\mathbf{CP}^n;\mathbf{Z}) \to \mathrm{H}^i(\mathbf{CP}^n;\mathbf{Z})$ is the same as the trace of $f_*\colon \mathrm{H}_i(\mathbf{CP}^n;\mathbf{Z}) \to \mathrm{H}_i(\mathbf{CP}^n;\mathbf{Z})$ for any map $f\colon \mathbf{CP}^n \to \mathbf{CP}^n$.

Given such a map, the induced map $f^*\colon \mathrm{H}^0(\mathbf{CP}^n;\mathbf{Z}) \to \mathrm{H}^0(\mathbf{CP}^n;\mathbf{Z})$ has trace 1. The induced map on the second cohomology groups is $x \mapsto ax$ for some $a \in \mathbf{Z}$. By naturality of cup product, this means that the map on the $2i$th cohomology groups is $x^i \mapsto a^i x^i$. Collecting these remarks, the Lefschetz number is

$$\tau(f) = \sum_{i=0}^{n} (-1)^{2i} a^i = \begin{cases} \frac{1-a^{n+1}}{1-a}, & a \neq 1 \\ n+1, & a = 1 \end{cases}.$$

This last number is nonzero unless $a = -1$ and n is odd. In particular, we have shown that if n is even, then f must have a fixed point by the Lefschetz fixed point theorem, and for n odd, we have shown the same except in the case that $f^*(x) = -x$ where x is the generator of $\mathrm{H}^2(\mathbf{CP}^n;\mathbf{Z})$. □

11. **Exercise.** Using cup products, show that every map $\mathrm{S}^{k+\ell} \to \mathrm{S}^k \times \mathrm{S}^\ell$ induces a trivial homomorphism $\mathrm{H}_{k+\ell}(\mathrm{S}^{k+\ell}) \to \mathrm{H}_{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell)$, assuming $k > 0$ and $\ell > 0$.

Solution. By the Künneth formula, $\mathrm{H}^*(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z}) \cong \mathrm{H}^*(\mathrm{S}^k;\mathbf{Z}) \otimes \mathrm{H}^*(\mathrm{S}^\ell;\mathbf{Z})$. In particular, the kth and ℓth cohomology groups of $\mathrm{S}^k \times \mathrm{S}^\ell$ are $\mathbf{Z}$ because $\mathrm{H}^k(\mathrm{S}^k;\mathbf{Z}) = \mathrm{H}^\ell(\mathrm{S}^\ell;\mathbf{Z}) = \mathbf{Z}$ by the universal coefficient theorem. On the other hand, the kth and ℓth cohomology groups of $\mathrm{S}^{k+\ell}$ are trivial. Thus, any map $f\colon \mathrm{S}^{k+\ell} \to \mathrm{S}^k \times \mathrm{S}^\ell$ induces the zero map on the kth and ℓth cohomology groups. Any element of $\mathrm{H}^{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z})$ can be written as a product of elements in $\mathrm{H}^k(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z})$ and $\mathrm{H}^\ell(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z})$, so the induced map $f^*\colon \mathrm{H}^{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z}) \to \mathrm{H}^{k+\ell}(\mathrm{S}^{k+\ell};\mathbf{Z})$ is also zero by naturality of cup product. Finally, by the naturality of the universal coefficient theorem (Hatcher p.196), the following diagram

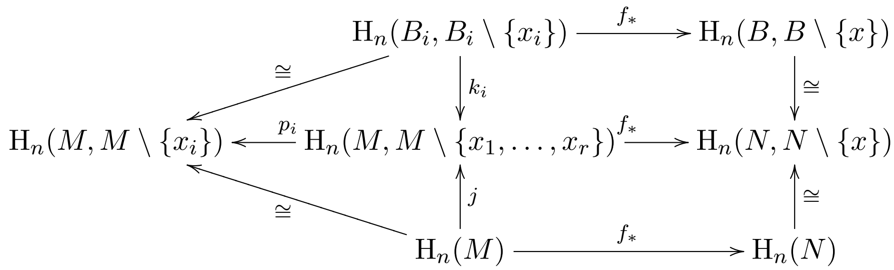
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}_{k+\ell-1}(\mathrm{S}^{k+\ell};\mathbf{Z}),\mathbf{Z}) & \longrightarrow & \mathrm{H}^{k+\ell}(\mathrm{S}^{k+\ell};\mathbf{Z}) & \xrightarrow{\varphi} & \mathrm{Hom}(\mathrm{H}_{k+\ell}(\mathrm{S}^{k+\ell};\mathbf{Z}),\mathbf{Z}) & \longrightarrow & 0 \\
 & & \uparrow (f_*)^* & & \uparrow f^* & & \uparrow (f_*)^* & & \\
0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}_{k+\ell-1}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z}),\mathbf{Z}) & \longrightarrow & \mathrm{H}^{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z}) & \xrightarrow{\varphi} & \mathrm{Hom}(\mathrm{H}_{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z}),\mathbf{Z}) & \longrightarrow & 0
\end{array}$$

commutes and the horizontal rows are exact. Since f^* is the zero map the surjectivity of the maps φ imply that the $(f_*)^*$ on the right is also zero. This map is the dual of the map $f_*\colon \mathrm{H}_{k+\ell}(\mathrm{S}^{k+\ell};\mathbf{Z}) \to \mathrm{H}_{k+\ell}(\mathrm{S}^k \times \mathrm{S}^\ell;\mathbf{Z})$, and hence f_* is also the zero map. □

3.3. Poincaré Duality

8. **Exercise.** For a map $f\colon M \to N$ between connected closed orientable n-manifolds, suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \varepsilon_i$ where ε_i is $+1$ or -1 according to whether $f\colon B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

Solution. Let x be a point in the interior of B, and let x_i be the point in B_i that maps to x. Also, let r be the number of balls B_i. Similar to the discussion of the degree of a map from a sphere to itself on p.136 of Hatcher, we have that the following diagram

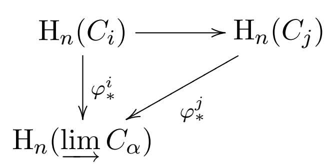


commutes, where the k_i and p_i are induced by inclusions. Taking the generator $[M] \in H_n(M)$, we know that $p_i j([M]) = \mu_{x_i}$, the local orientation at x_i, by the commutativity of the lower triangle. By excision, the middle term $H_n(M, M \setminus \{x_1, \ldots, x_r\})$ is isomorphic to the direct sum of the groups $H_n(B_i, B_i \setminus \{x_i\}) \cong \mathbf{Z}$, and k_i is the inclusion map into the ith summand. Since the p_i is projection onto the ith summand, we see that $j([M]) = \sum_{i=1}^{r} k_i(\mu_{x_i})$ by commutativity of the upper triangle.

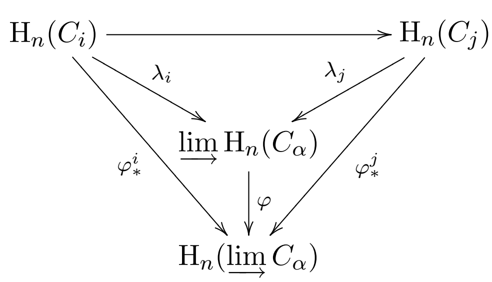
By the commutativity of the upper square, we deduce that $f_*(k_i(\mu_{x_i})) = \varepsilon_i$ where $\varepsilon_i = \pm 1$ depending on whether f preserves or reverses the orientation of B_i when mapping to B. Finally, by commutativity of the lower square, we conclude that $f_*([M]) = (\sum_{i=1}^{r} f_*(k_i(\mu_{x_i})))[N] = (\sum_{i=1}^{r} \varepsilon_i)[N]$. Thus, $\deg f = \sum_{i=1}^{r} \varepsilon_i$. □

17. **Exercise.** Show that a direct limit of exact sequences is exact. More generally, show that homology commutes with direct limits: If $\{C_\alpha, f_{\alpha\beta}\}$ is a directed system of chain complexes, with the maps $f_{\alpha\beta}\colon C_\alpha \to C_\beta$ chain maps, then $H_n(\varinjlim C_\alpha) = \varinjlim H_n(C_\alpha)$.

Solution. There is a canonical map $\varphi^i\colon C_i \to \varinjlim C_\alpha$ for all i which induces a map on homology $H_n(C_i) \to H_n(\varinjlim C_\alpha)$. By the functoriality of homology, these induced maps are compatible with the maps $H_n(C_i) \to H_n(C_j)$, i.e., the following diagram



commutes for all i and j for which there is a map f_{ij}. By the universal property of direct limit, this induces a map $\varphi\colon \varinjlim H_n(C_\alpha) \to H_n(\varinjlim C_\alpha)$ such that the following diagram



commutes for all i and j for which there is a map f_{ij} and where λ_i denotes the canonical map into a direct limit. We claim that φ is an isomorphism.

For surjectivity, choose $x \in H_n(\varinjlim C_\alpha)$. Then x is a cycle, and hence $\partial x = 0$ where ∂ is the differential in $\varinjlim C_\alpha$. Pick a representative $x_i \in C_i$ of x, i.e., $\varphi^i(x_i) = x$. Then $\varphi^i_*(\partial_i x_i) = 0$, and hence there exists some j such that $f_{ij}(\partial_i x_i) = 0$. This means that $f_{ij}(x_i) \in \mathrm{H}_n(C_j)$. Setting $y = \lambda_j(f_{ij}(x_i))$, we have $\varphi(y) = x$.

To see injectivity, suppose $x \in \varinjlim \mathrm{H}_n(C_\alpha)$ is mapped to 0 by φ. Choose a representative $x_i \in \mathrm{H}_n(C_i)$ of x, i.e., $\lambda_i(x_i) = x$. Then $\varphi^i_*(x_i)$ is a boundary of some element, i.e., there exists y such that $\partial y = \varphi^i_*(x_i)$ where ∂ is the differential of $\varinjlim C_\alpha$. Then we can find a representative $y_j \in C_j$ of y for some j with $\partial_j y_j = 0$. But $\partial_j y_j$ is also a representative of x, so $x = 0$.

Therefore, φ is an isomorphism, so direct limits commute with homology. The statement about direct limits preserving exact sequences follows because exactness is equivalent to homology being trivial. □

32. **Exercise.** Show that a compact manifold does not retract onto its boundary.

    Solution. Let M be a compact manifold and suppose that there is a retraction $r\colon M \to \partial M$. Let $i\colon \partial M \hookrightarrow M$ be the inclusion, so that $r \circ i$ is the identity on M. This implies that the induced map on homology $i_*\colon \mathrm{H}_{n-1}(\partial M; \mathbf{Z}/2) \to \mathrm{H}_{n-1}(M; \mathbf{Z}/2)$ is injective by functoriality of homology. By the long exact sequence of relative homology,

$$\mathrm{H}_n(M, \partial M; \mathbf{Z}/2) \xrightarrow{\partial} \mathrm{H}_{n-1}(\partial M; \mathbf{Z}/2) \xrightarrow{i_*} \mathrm{H}_{n-1}(M; \mathbf{Z}/2)$$

    is exact. This implies that $\partial = 0$ because i_* is injective. But this contradicts exercise 3.3.31 of Hatcher, which says that ∂ sends a fundamental class of $(M, \partial M)$ to a fundamental class of ∂M. Thus, M does not retract onto its boundary. □

3.C. H-Spaces and Hopf Algebras

5. **Exercise.** Show that if (X, e) is an H-space then $\pi_1(X, e)$ is Abelian.

   Solution. Choose $f, f', g, g' \in \pi_1(X, e)$ and suppose that H is a homotopy $f \simeq f'$ and H' is a homotopy $g \simeq g'$. We claim that $H * H'\colon I \times I \to X$ defined by $(s,t) \mapsto H(s,t) * H'(s,t)$ is a homotopy $f * g \simeq f' * g'$ where $f * g\colon I \to X$ is defined by $s \mapsto f(s) * g(s)$. Indeed, $(H*H')(0,t) = H(0,t)*H'(0,t) = e*e = e$, and similarly, $(H*H')(1,t) = e$. Also, $(H*H')(s,0) = H(s,0) * H'(s,0) = f(s) * g(s)$ and similarly $(H * H')(s,1) = f'(s) * g'(s)$. Since $H * H'$ is the composition of continuous maps, it is continuous, and thus the desired homotopy. Now pick any $h, h' \in \pi_1(X, e)$. Let 1 denote the constant path with base point e. Then $h * h' \simeq (h \cdot 1) * (1 \cdot h')$. Since $(h \cdot 1)(s)$ is $h(2s)$ if $0 \le s \le 1/2$ and is e otherwise, and $(1 \cdot h')(s)$ is $h'(2s-1)$ if $1/2 \le s \le 1$ and e otherwise, we get $(h \cdot 1) * (1 \cdot h') \simeq h \cdot h'$. By the same reasoning, $h \cdot h' \simeq (1 \cdot h') * (h \cdot 1)$, and this is homotopic to $h' \cdot h$. This gives that $h \cdot h' \simeq h' \cdot h$, so $\pi_1(X, e)$ is Abelian. □

4 Homotopy Theory

4.1. Homotopy Groups

2. **Exercise.** Show that if $\varphi\colon X \to Y$ is a homotopy equivalence, then the induced homomorphisms $\varphi_*\colon \pi_n(X, x_0) \to \pi_n(Y, \varphi(x_0))$ are isomorphisms for all n.

   Solution. First, the technique of p.341 shows an analogue of Lemma 1.19 for higher homotopy groups. Let $\psi\colon Y \to X$ be a homotopy inverse of φ. Then $\psi\varphi \simeq 1_X$ implies that $\psi_*\varphi_*$ is an isomorphism because it is equal to a change of base point isomorphism as described in p.341. Similarly, $\varphi_*\psi_*$ is an isomorphism, so we conclude that φ_* is an isomorphism for all n. □

3. **Exercise.** For an H-space (X, x_0) with multiplication $\mu\colon X \times X \to X$, show that the group operation in $\pi_n(X, x_0)$ can also be defined by the rule $(f+g)(x) = \mu(f(x), g(x))$.

Solution. Writing $x = (x_1, \dots, x_n) \in I^n$, the sum on the left-hand side is defined to be

$$(f+g)(x_1, \dots, x_n) = \begin{cases} f(2x_1, x_2, \dots, x_n), & x_1 \in [0, 1/2] \\ g(2x_1 - 1, x_2, \dots, x_n), & x_1 \in [1/2, 1] \end{cases}.$$

Letting 1 denote the constant map $I^n \to x_0$, we have $f + g$ is homotopic to

$$\begin{cases} \mu(f(2x_1, x_2, \dots, x_n), 1), & x_1 \in [0, 1/2] \\ \mu(1, g(2x_1 - 1, x_2, \dots, x_n)), & x_1 \in [1/2, 1] \end{cases},$$

which in turn is homotopic to

$$\mu(f(x_1, x_2, \dots, x_n), g(x_1, x_2, \dots, x_n)). \quad \square$$

5. **Exercise.** For a pair (X, A) of path-connected spaces, show that $\pi_1(X, A, x_0)$ can be identified in a natural way with the set of cosets αH of the subgroup $H \subset \pi_1(X, x_0)$ represented by loops in A at x_0.

Solution. By definition, $\pi_1(X, A, x_0)$ is the set of homotopy classes of paths in X from a varying point in A to $x_0 \in A$. Define a map $\pi_1(X, x_0) \to \pi_1(X, A, x_0)$ by thinking of a loop at x_0 as an element of $\pi_1(X, A, x_0)$. Since A is path-connected, every element of $\pi_1(X, A, x_0)$ is homotopic to a loop at x_0, so this map is surjective. Note that two loops $\gamma_0, \gamma_1 \in \pi_1(X, x_0)$ are homotopic relative to A if and only if $\gamma_0^{-1}\gamma_1$ is represented by a loop in A, so we can identify $\pi_1(X, A, x_0)$ with the set of cosets αH. $\square$

10. **Exercise.** Show that the 'quasi-circle' described in (Ex. 1.3.7) has trivial homotopy groups but is not contractible, hence does not have the homotopy type of a CW complex.

Solution. Let Y be the quasi-circle. Since Y has infinite length, the image of any map $I^n \to Y$ must live in some region homeomorphic to the unit interval, or the disjoint union of two copies of the unit interval with their midpoints identified. Both such spaces are contractible, so Y has trivial homotopy groups. However, this space is not contractible, because identifying the part of the graph of $y = \sin(1/x)$ to a single point gives the circle. $\square$

11. **Exercise.** Show that a CW complex is contractible if it is the union of an increasing sequence of subcomplexes $X_1 \subset X_2 \subset \cdots$ such that each inclusion $X_i \hookrightarrow X_{i+1}$ is nullhomotopic, a condition sometimes expressed by saying X_i is contractible in X_{i+1}. An example is S^∞, or more generally the infinite suspension $S^\infty X$ of any CW complex X, the union of the iterated suspensions $S^n X$.

Solution. By Whitehead's theorem, it is enough to show that all of the homotopy groups of $X = \bigcup_{i \geq 0} X_i$ are trivial to show that it is contractible. Let $\varphi\colon \mathrm{S}^n \to X$ be a map. By cellular approximation, we may assume that is cellular. Since the image of φ is compact, it intersects finitely many n-cells of X, so the image lives inside some X_k. Since $X_k \hookrightarrow X_{k+1} \hookrightarrow X$ is nullhomotopic, φ is also nullhomotopic, so $\pi_n(X) = 0$. $\square$

12. **Exercise.** Use the extension lemma to show that a CW complex retracts onto any contractible subcomplex.

Solution. Let X be a CW complex with a contractible subcomplex A. Since A is contractible, it is path-connected. The identity map $A \to A$ can be extended to a map $X \to A$ since all of the homotopy groups of A are trivial (Lemma 4.7). This extension is the desired retraction. $\square$

13. **Exercise.** Use cellular approximation to show that the n-skeletons of homotopy equivalent CW complexes without cells of dimension $n+1$ are also homotopy equivalent.

    Solution. Let $f\colon X \to Y$ be a homotopy equivalence between two CW complexes X and Y with homotopy inverse g. By cellular approximation, we may assume that both f and g are cellular maps, i.e., define maps $X^n \to Y^n$ and $Y^n \to X^n$. The homotopy $h\colon X \times [0,1] \to X$ from f to g can also be replaced by a cellular map, the image lies inside of $X^{n+1} = X^n$ since X has no cells of dimension $n+1$. Hence gf restricted to X^n is homotopic to the identity map. Similarly, fg restricted to Y^n is homotopic to the identity map, so $X^n \simeq Y^n$. □

14. **Exercise.** Show that every map $f\colon \mathrm{S}^n \to \mathrm{S}^n$ is homotopic to a multiple of the identity map by the following steps.

    (a) Reduce to the case that there exists a point $q \in \mathrm{S}^n$ with $f^{-1}(q) = \{p_1, \dots, p_k\}$ and f is an invertible map near each p_i.

    (b) For f as in (a), consider the composition gf where $g\colon \mathrm{S}^n \to \mathrm{S}^n$ collapses the complement of a small ball about q to the basepoint. Use this to reduce (a) further to the case $k = 1$.

    (c) Finish the argument by showing that an invertible $n \times n$ matrix can be joined by a path of such matrices to either the identity matrix or the matrix of a reflection.

    Solution. By Theorem 2C.1, f is homotopic to a map that is simplicial with respect to some iterated barycentric subdivision of S^n. Hence there exists a point $q \in \mathrm{S}^n$ such that $f^{-1}(q) = \{p_1, \dots, p_k\}$, and f is a piecewise linear map around each p_i and hence is locally invertible. For each i, we can intersect the images of the neighborhoods around p_i on which f is invertible to get a small ball around q. Let g be the map which collapses the complement of this small ball to the basepoint. For each i, we then get a map $f_i\colon \mathrm{S}^n \to \mathrm{S}^n$ such that $f_i^{-1}(q) = \{p_i\}$ by identifying the neighborhood around p_i with S^n (by collapsing its boundary to a point) and letting f_i be the restriction of gf. Then f is homotopic to the sum of the f_i, so we can reduce to the case $k = 1$. Thinking of p and q as points at infinity, and using the fact that f is linear, we can think of $\mathrm{S}^n \setminus p \to \mathrm{S}^n \setminus q$ as an invertible $n \times n$ matrix. Using Gaussian elimination, we can find a piecewise linear path from such a matrix either to the identity matrix, or the matrix of a reflection, depending on the sign of its determinant. Such a path gives a homotopy of f either to the identity map or the reflection, which is -1 times the identity map. □

19. **Exercise.** Consider the equivalence relation $\simeq_w$ generated by weak homotopy equivalence: $X \simeq_w Y$ if there are spaces $X = X_1, X_2, \dots, X_n = Y$ with weak homotopy equivalences $X_i \to X_{i+1}$ or $X_i \leftarrow X_{i+1}$ for each i. Show that $X \simeq_w Y$ if and only if X and Y have a common CW approximation.

    Solution. A CW approximation of X comes with a weak homotopy equivalence, so if X and Y have a common CW approximation, then $X \simeq_w Y$ by definition. Conversely, suppose that $X \simeq_w Y$. We wish to show that X and Y have a common CW approximation. Without loss of generality, we may assume that we have a weak homotopy equivalence $g\colon X \to Y$. Let X' and Y' be CW approximations for X and Y. Then by Proposition 4.18, there exists a map $h\colon X' \to Y'$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f_1} & X \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow{f_2} & Y \end{array}$$

commutes up to homotopy. On homotopy groups, f_{1*}, g, and f_{2*} are isomorphisms, so h_* is also an isomorphism. By Whitehead's theorem, h is a homotopy equivalence, so X and Y have a common CW approximation. □

20. **Exercise.** Show that $[X, Y]$ is finite if X is a finite connected CW complex and $\pi_i(Y)$ is finite for $i \leq \dim X$.

Solution. Given a map $f\colon X \to Y$ and a cell X' of X, there are only finitely many maps $X' \to X$ up to homotopy that the restriction $f|_{X'}$ could be because one can think of this map as a composition of the attaching map for X' with f. Hence, there are only finitely many maps f up to homotopy because the homotopies on the individual cells are relative to their boundary, so this shows that they determine f up to homotopy. Hence $[X, Y]$ is finite. □

4.2. Elementary Methods of Calculation

1. **Exercise.** Use homotopy groups to show there is no retraction $\mathbf{RP}^n \to \mathbf{RP}^k$ if $n > k > 0$.

Solution. The quotient map $\mathrm{S}^n \to \mathbf{RP}^n$ is a covering space whose fiber F consists of two points with the discrete topology. Hence we have isomorphisms $\pi_i(\mathrm{S}^n) \cong \pi_i(R\mathbf{P}^n)$ (Proposition 4.1) for $i > 1$. In particular, if there were a retraction $r\colon \mathbf{RP}^n \to \mathbf{RP}^k$, then there is a map $i\colon \mathbf{RP}^k \to \mathbf{RP}^n$ such that $r \circ i$ is the identity on $\mathbf{RP}^k$. On homotopy groups, this becomes the fact that $\pi_i(\mathbf{RP}^k) \to \pi_i(\mathbf{RP}^n) \to \pi_i(\mathbf{RP}^k)$ is the identity map. In particular, if $i = k$, then $\pi_k(\mathbf{RP}^k) = \mathbf{Z}$ and $\pi_k(\mathbf{RP}^n) = 0$ for $n > k$, so this is a contradiction. □

4. **Exercise.** Let $X \subset \mathbf{R}^{n+1}$ be the union of the infinite sequence of spheres S^n_k of radius $\frac{1}{k}$ and center $(\frac{1}{k}, 0, \ldots, 0)$. Show that $\pi_i(X) = 0$ for $i < n$ and construct a homomorphism from $\pi_n(X)$ onto $\prod_k \pi_n(\mathrm{S}^n_k)$.

Solution. Since S^i is compact, the image of any map $\mathrm{S}^i \to X$ can only intersect finitely many of the S^n_k, so if $i < n$, the images on each such S^n_k can be homotoped to the origin, and hence is homotopic to a constant map. So $\pi_i(X) = 0$ for $i < n$. We can divide the cube I^n into parts $I^n_k = \{(x_1, \ldots, x_n) \in I^n \mid 2^{-k} \leq x_1 \leq 2^{-k-1}\}$ for $k \geq 0$. Define a map $X \to \bigvee_k \mathrm{S}^n_k$ by choosing a basepoint for each S^n_k. Then compose this with the inclusion $\bigvee_k \mathrm{S}^n_k \to \prod_k \mathrm{S}^n_k$. Call the induced map on homotopy groups $p\colon \pi_n(X) \to \prod_k \pi_n(\mathrm{S}^n_k)$. An element of $\prod_k \pi_n(\mathrm{S}^n_k)$ is a sequence of integers $(a_1, a_2, \ldots)$. Define a map $f\colon I^n \to X$ by defining the restriction $f\colon I^n_k \to \mathrm{S}^n_k$ to be a degree a_k map (here we are identifying I^n_k with I^n via some homeomorphism that preserves the boundaries). Then $p(f) = (a_1, a_2, \ldots)$, so p is surjective. □

6. **Exercise.** Show that the relative form of the Hurewicz theorem in dimension n implies the absolute form in dimension $n-1$ by considering the pair (CX, X) where CX is the cone on X.

Solution. Let X be a $(n-1)$-connected space. Since CX is contractible, by the long exact sequence of homotopy groups of a pair, we see that (CX, X) is n-connected, and $\pi_{n-1}(X) \cong \pi_n(CX, X)$. So by the relative Hurewicz, $\mathrm{H}_i(CX, X) = 0$ for $i < n$ and $\pi_n(CX, X) \cong \mathrm{H}_n(CX, X)$. Now by the long exact sequence on homology for a pair, $\mathrm{H}_i(X) \cong \mathrm{H}_{i+1}(CX, X)$, hence we deduce the absolute Hurewicz in dimension $n-1$. □

8. **Exercise.** Show the suspension of an acyclic CW complex is contractible.

Solution. Let X be an acyclic CW complex, i.e., $\widetilde{\mathrm{H}}_i(X) = 0$ for all i. This means that X is a connected space. By the Freudenthal suspension theorem (Corollary 4.24), we have an isomorphism $\pi_0(X) \cong \pi_1(SX)$, and a surjection $\pi_1(X) \to \pi_2(SX)$. Since the Abelianization of $\pi_1(X)$ is $\mathrm{H}_1(X) = 0$, and $\pi_2(SX)$ is Abelian, this implies that $\pi_2(SX) = 0$, so SX is 2-connected. We claim that $\widetilde{\mathrm{H}}_i(X) \cong \widetilde{\mathrm{H}}_{i+1}(SX)$ for all i, so SX is also an acyclic CW complex. To see this, by Proposition 2.22, we have $\mathrm{H}_i(X \times [0,1], X \times \{0,1\}) \cong \widetilde{\mathrm{H}}_i(SX)$ for all i. Then we have the desired isomorphism by considering the long exact sequence on homology for the pair $(X \times [0,1], X \times \{0,1\})$: the map $\mathrm{H}_i(X \times \{0,1\}) \to \mathrm{H}_i(X \times [0,1])$ can be thought of as $\mathrm{H}_i(X) \oplus \mathrm{H}_i(X) \to \mathrm{H}_i(X \times [0,1])$ with inclusion for each factor. So this map is surjective, and its kernel is the subgroup generated by $(x, -x)$, so is isomorphic to $\mathrm{H}_i(X)$. So this shows the claim. Then by the Hurewicz theorem, we must have that $\pi_i(SX) = 0$ for all i, which means that SX is contractible by Whitehead's theorem. □

9. **Exercise.** Show that a map between simply-connected CW complexes is a homotopy equivalence if its mapping cone is contractible. Use the preceding exercise to give an example where this fails in the nonsimply-connected case.

   Solution. Let $f\colon X \to Y$ be a map of simply-connected CW complexes, and let M_f be its mapping cylinder with mapping cone M_f/X. Note that M_f is simply-connected since it is homotopy equivalent to Y. Suppose that M_f/X is contractible. Then the inclusion $X \to M_f$ induces isomorphisms on homology $\mathrm{H}_n(X) \to \mathrm{H}_n(M_f)$ for all n, so is a homotopy equivalence (Corollary 4.33). This implies that f is also a homotopy equivalence.

   Now let X be a noncontractible acyclic CW complex (Example 2.38). Then the suspension map $f\colon X \to SX$ gives rise to a contractible mapping cylinder since SX is contractible (Ex. 4.2.8), but f cannot be a homotopy equivalence. □

10. **Exercise.** Let the CW complex X be obtained from $\mathrm{S}^1 \vee \mathrm{S}^n$, $n \geq 2$, by attaching a cell e^{n+1} by a map representing the polynomial $p(t) \in \mathbf{Z}[t, t^{-1}] \cong \pi_n(\mathrm{S}^1 \vee \mathrm{S}^n)$, so $\pi_n(X) \cong \mathbf{Z}[t, t^{-1}]/(p(t))$. Show $\pi'_n(X)$ is cyclic and compute its order in terms of $p(t)$. Give examples showing that the group $\pi_n(X)$ can be finitely generated or not, independently of whether $\pi'_n(X)$ is finite or infinite.

    Solution. Since $\pi_n(\mathrm{S}^1 \vee \mathrm{S}^n)$ is a free $\mathbf{Z}[t, t^{-1}]$-module on one generator, the map $\pi_n(X) \to \pi'_n(X)$ is obtained by substituting 1 for t. So the relation $p(t) = 0$ in $\pi'_n(X)$ becomes $\sum a_i = 0$ where the a_i are the coefficients of p. So $\pi'_n(X)$ is cyclic of infinite order if $\sum a_i = 0$, and is cyclic of finite order c if $\sum a_i = c \neq 0$.

    If for example $p(t) = 0$, then $\pi_n(X) = \mathbf{Z}[t, t^{-1}]$ is not finitely generated, and $\pi'_n(X) = \mathbf{Z}$ is infinite. On the other hand, if $p(t) = t - 1$, then $\pi_n(X) \cong \mathbf{Z}$ is finitely generated, but $\pi'_n(X)$ is infinite. If $p(t) = t$, then $\pi_n(X) = 0$ is finitely generated, and $\pi'_n(X) = 0$ is finite. Finally, if $p(t) = t^2 + t + 1$, then $\pi_n(X)$ is not finitely generated since $\{t^{-1}, t^{-2}, \dots\}$ has no finite generating set, but $\pi'_n(X) = \mathbf{Z}/3$ is finite. □

12. **Exercise.** Show that a map $f\colon X \to Y$ of connected CW complexes is a homotopy equivalence if it induces an isomorphism on π_1 and if a lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ to the universal covers induces an isomorphism on homology.

    Solution. The commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

where the vertical maps are the covering maps, induces a commutative diagram

$$\begin{array}{ccc} \pi_i(\widetilde{X}) & \xrightarrow{\widetilde{f}_*} & \pi_i(\widetilde{Y}) \\ \downarrow & & \downarrow \\ \pi_i(X) & \xrightarrow{f_*} & \pi_i(Y). \end{array}$$

The vertical maps are isomorphisms for $i > 1$ (Proposition 4.1). Since $\widetilde{X}$ and $\widetilde{Y}$ are simply-connected CW complexes, $\widetilde{f}$ is a homotopy equivalence (Corollary 4.33). Hence $\widetilde{f}_*$ is an isomorphism for all i, which implies that f_* is an isomorphism for all $i > 1$ by the commutativity of the diagram. By assumption, f_* is also an isomorphism for $i = 1$, so f is a homotopy equivalence. □

13. **Exercise.** Show that a map between connected n-dimensional CW complexes is a homotopy equivalence if it induces an isomorphism on π_i for $i \leq n$.

    Solution. Let $f\colon X \to Y$ be a map between connected n-dimensional CW complexes which induces an isomorphism on π_i for $i \leq n$. Passing to universal covers, and taking a lift $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$, we get a map which induces isomorphisms on π_i for $i \leq n$. By the Hurewicz theorem, this implies that $\widetilde{f}$ is an isomorphism $\mathrm{H}_i(\widetilde{X}) \to \mathrm{H}_i(\widetilde{Y})$ for $i \leq n$. It is also an isomorphism for $i > n$ since the homology vanishes in these degrees (this can be seen by cellular homology). Hence f is a homotopy equivalence (Ex. 4.2.12). □

15. **Exercise.** Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3.

    Solution. Let M be a closed simply-connected 3-manifold. First, M is homotopy equivalent to a CW complex. Second, M is orientable since it is simply-connected (otherwise, the orientation covering would be connected). Since $\pi_1(M) = 0$, we have $\mathrm{H}_1(M) = 0$, so $\mathrm{H}_2(M) = 0$ by Poincaré duality, and the top homology is $\mathrm{H}_3(M) = \mathbf{Z}$. Now let $f\colon \mathrm{S}^3 \to M$ be a map of degree 1. This exists because $\pi_3(M) = \mathbf{Z}$ by the Hurewicz theorem. Then f induces isomorphisms on homology, so is a homotopy equivalence because M and S^3 are simply-connected. □

16. **Exercise.** Show that the closed surfaces with infinite fundamental group are $K(\pi, 1)$'s by showing that their universal covers are contractible, via the Hurewicz theorem and results of section 3.3.

    Solution. Let X be a closed surface with infinite fundamental group, and let $\widetilde{X}$ be its universal cover. Since $\pi_1(X)$ is infinite, $\widetilde{X}$ is not compact. Hence, $\mathrm{H}_2(\widetilde{X}) = 0$ (Proposition 3.29). Also, $\mathrm{H}_1(\widetilde{X}) = 0$ since $\pi_1(\widetilde{X}) = 0$, so $\widetilde{X}$ is contractible since it has the homotopy type of a CW complex. Hence X is a $K(\pi_1(X), 1)$. □

18. **Exercise.** If X and Y are simply-connected CW complexes such that $\widetilde{\mathrm{H}}_i(X)$ and $\widetilde{\mathrm{H}}_j(Y)$ are finite and of relatively prime orders for all pairs (i, j), show that the inclusion $X \vee Y \hookrightarrow X \times Y$ is a homotopy equivalence and $X \wedge Y$ is contractible.

    Solution. By the Künneth formula, $\widetilde{\mathrm{H}}_n(X \times Y) \cong \widetilde{\mathrm{H}}_n(X) \oplus \widetilde{\mathrm{H}}_n(Y)$. Since the is the image of the map $\widetilde{\mathrm{H}}_n(X \vee Y) \to \widetilde{\mathrm{H}}_n(X \times Y)$ induced by the inclusion, this is an isomorphism. Hence $X \vee Y \hookrightarrow X \times Y$ is a homotopy equivalence. Also, $X \wedge Y$ is contractible since $\widetilde{\mathrm{H}}_n(X \wedge Y) = 0$ by the long exact sequence of the pair $(X \times Y, X \vee Y)$ and the fact that $X \times Y / X \vee Y = X \wedge Y$. □

20. **Exercise.** Let G be a group and X a simply-connected space. Show that for the product $K(G, 1) \times X$ the action of π_1 on π_n is trivial for all $n > 1$.

Solution. An element $f \in \pi_n$ is represented by maps $f_1 \colon I^n \to K(G,1)$ and $f_2 \colon I^n \to X$ which map the boundary to fixed basepoints, and similarly, an element of $\gamma \in \pi_1$ is represented by maps $\gamma_1 \colon [0,1] \to K(G,1)$ and $\gamma_2 \colon [0,1] \to X$ which map the boundary to the basepoints. The action of γ on f is then obtained by shrinking the domain of I^n for f_i homeomorphically and inserting γ_i into the remainder of I^n. Since $\pi_n(K(G,1)) = 0$ for $n > 1$, $\gamma_1 f_1$ is homotopic to f_1. Since γ_2 is homotopic to a constant map, $\gamma_2 f_2$ is also homotopic to f_2, so γf is homotopic to f. □

21. **Exercise.** Given a sequence of CW complexes $K(G_n, n)$, $n = 1, 2, \dots$, let X_n be the CW complex formed by the product of the first n of these $K(G_n, n)$'s. Via the inclusions $X_{n-1} \subset X_n$ coming from regarding X_{n-1} as the subcomplex of X_n with nth coordinate equal to a basepoint 0-cell of $K(G_n, n)$, we can then form the union of all the X_n's, a CW complex X. Show $\pi_n(X) \cong G_n$ for all n.

Solution. Let $f \colon \mathrm{S}^n \to X$ be a map. By compactness, the image of S^n must lie inside of some X_m. If $m < n$, then f is homotopic to a constant map, otherwise, we have $\pi_n(X_m) \cong G_n$. □

22. **Exercise.** Show that $\mathrm{H}_{n+1}(K(G,n); \mathbf{Z}) = 0$ if $n > 1$.

Solution. Let $M(G,n)$ be a Moore space. Since $M(G,n)$ is simply-connected for $n > 1$, we can use the Hurewicz theorem to deduce that $M(G,n)$ is $(n-1)$-connected. To turn $M(G,n)$ into a $K(G,n)$, we can attach cells of dimensions $\geq n+2$ to kill the higher homotopy groups. Doing this does not affect the homology in degrees $\leq n+1$, so we conclude that $\mathrm{H}_{n+1}(K(G,n)) = \mathrm{H}_{n+1}(M(G,n)) = 0$. □

23. **Exercise.** Extend the Hurewicz theorem by showing that if X is an $(n-1)$-connected CW complex, then the Hurewicz homomorphism $h \colon \pi_{n+1}(X) \to \mathrm{H}_{n+1}(X)$ is surjective when $n > 1$, and when $n = 1$ show there is an isomorphism $\mathrm{H}_2(X)/h(\pi_2(X)) \cong \mathrm{H}_2(K(\pi_1(X), 1))$.

Solution. First, we can build a $K(\pi_n(X), n)$ from X by attaching cells of dimension $\geq n+2$. Let Y be the result of attaching all of the $(n+2)$-cells of $K(\pi_n(X), n)$ to X. From the naturality of the Hurewicz homomorphism, the diagram

$$\begin{array}{ccccccc}
\pi_{n+2}(Y,X) & \longrightarrow & \pi_{n+1}(X) & \longrightarrow & \pi_{n+1}(Y) & \longrightarrow & \pi_{n+1}(Y,X) \\
\downarrow & & \downarrow h & & \downarrow & & \downarrow \\
\mathrm{H}_{n+2}(Y,X) & \xrightarrow{\partial} & \mathrm{H}_{n+1}(X) & \longrightarrow & \mathrm{H}_{n+1}(Y) & \longrightarrow & 0
\end{array}$$

commutes. That $\mathrm{H}_{n+1}(Y,X) = 0$ comes from the fact that Y and X have the same $(n+1)$-cells, and hence $C_{n+1}(Y,X) = 0$. From the definition of the Hurewicz homomorphism, the images of $h \colon \pi_{n+1}(X) \to \mathrm{H}_{n+1}(X)$ and $\partial \colon \mathrm{H}_{n+2}(Y,X) \to \mathrm{H}_{n+1}(X)$ coincide since ∂ sends a relative cycle in Y to its boundary in X. Note also that $\mathrm{H}_{n+1}(Y) \cong \mathrm{H}_{n+1}(K(\pi_n(X), n))$ by cellular homology. So from the above, we get $\mathrm{H}_{n+1}(X)/h(\pi_{n+1}(X)) \cong \mathrm{H}_{n+1}(K(\pi_n(X), n))$. Hence if $n > 1$, then by (Ex. 4.2.22), $\mathrm{H}_{n+1}(Y) = 0$, so ∂, and hence h, is surjective. If $n = 1$, this becomes $\mathrm{H}_2(X)/h(\pi_2(X)) \cong \mathrm{H}_2(K(\pi_1(X), 1))$. □

26. **Exercise.** Generalizing the example of $\mathbf{RP}^2$ and $\mathrm{S}^2 \times \mathbf{RP}^\infty$, show that if X is a connected finite-dimensional CW complex with universal cover $\widetilde{X}$, then X and $\widetilde{X} \times K(\pi_1(X), 1)$ have isomorphic homotopy groups but are not homotopy equivalent if $\pi_1(X)$ contains elements of finite order.

Solution. It is immediate that X and $\widetilde{X} \times K(\pi_1(X), 1)$ have isomorphic homotopy groups since $\pi_i(\widetilde{X}) \cong \pi_i(X)$ for $i > 2$, and $\pi_1(\widetilde{X}) = 0$.

Suppose that $\pi_1(X)$ has elements of finite order. Then by Proposition 2.45, $K(\pi_1(X), 1)$ must be an infinite-dimensional CW complex. By the Künneth formula, $K(\pi_1(X), 1)$ has infinitely many nontrivial homology groups (for example, this homology agrees with the group homology of $\pi_1(X)$, and the group homology of finite cyclic groups satisfies this property), while X has only finitely many since it is finite-dimensional. Hence X and $\widetilde{X} \times K(\pi_1(X), 1)$ cannot be homotopy equivalent spaces. □

27. **Exercise.** From Lemma 4.39 deduce that the image of the map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ lies in the center of $\pi_2(X, A, x_0)$.

    Solution. By the long exact sequence of homotopy groups for a pair (X, A), the image of the map $\pi_2(X, x_0) \to \pi_2(X, A, x_0)$ is equal to the kernel of the boundary map $\partial\colon \pi_2(X, A, x_0) \to \pi_1(A, x_0)$. Pick $x \in \ker \partial$. Then $xbx^{-1} = e$ for all $b \in \pi_2(X, A, x_0)$ by Lemma 4.39, so x is in the center of $\pi_2(X, A, x_0)$. □

28. **Exercise.** Show that the group $\mathbf{Z}/p \times \mathbf{Z}/p$ with p prime cannot act freely on any sphere S^n, by filling in the details of the following argument. Such an action would define a covering space $\mathrm{S}^n \to M$ with M a closed manifold. When $n > 1$, build a $K(\mathbf{Z}/p \times \mathbf{Z}/p, 1)$ from M by attaching a single $(n+1)$-cell and then cells of higher dimension. Deduce that $\mathrm{H}^{n+1}(K(\mathbf{Z}/p \times \mathbf{Z}/p, 1); \mathbf{Z}/p)$ is $\mathbf{Z}/p$ or 0, a contradiction.

    Solution. Suppose $\mathbf{Z}/p \times \mathbf{Z}/p$ acts freely on S^n. Thinking of $\mathbf{Z}/p \times \mathbf{Z}/p$ as a 0-dimensional Lie group, we have a free, proper, smooth action on S^n, which gives a smooth submersion $\mathrm{S}^n \to M = \mathrm{S}^n/(\mathbf{Z}/p\times\mathbf{Z}/p)$. In particular, M is a smooth compact n-manifold. Since $\dim M = n$, this is a covering space map whose group of deck transformations is $\mathbf{Z}/p \times \mathbf{Z}/p$. In particular, $\pi_1(M) = \mathbf{Z}/p \times \mathbf{Z}/p$.

    If $n = 1$, then M must be homotopy equivalent to either S^1 or $[0, 1]$ since these are the only connected compact 1-manifolds, so we have a contradiction since their respective fundamental groups are $\mathbf{Z}$ and 0.

    If $n > 1$, then we can make M a $K(\mathbf{Z}/p \times \mathbf{Z}/p, 1)$ by attaching a single $(n + 1)$-cell and then attaching more cells of higher dimension. This shows that $\mathrm{H}^{n+1}(K(\mathbf{Z}/p \times \mathbf{Z}/p, 1); \mathbf{Z}/p)$ is either $\mathbf{Z}/p$ or 0 by cellular homology. But this contradicts the group cohomology of $\mathbf{Z}/p \times \mathbf{Z}/p$ (which is bigger than $\mathbf{Z}/p$ by the Künneth formula). □

31. **Exercise.** For a fiber bundle $F \to E \to B$ such that the inclusion $F \hookrightarrow E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $\mathrm{S}^3 \to \mathrm{S}^7 \to \mathrm{S}^4$ and $\mathrm{S}^7 \to \mathrm{S}^{15} \to \mathrm{S}^8$ this yields isomorphisms

$$\pi_n(\mathrm{S}^4) \cong \pi_n(\mathrm{S}^7) \oplus \pi_{n-1}(\mathrm{S}^3)$$
$$\pi_n(\mathrm{S}^8) \cong \pi_n(\mathrm{S}^{15}) \oplus \pi_{n-1}(\mathrm{S}^7)$$

    Thus $\pi_7(\mathrm{S}^4)$ and $\pi_{15}(\mathrm{S}^8)$ contain $\mathbf{Z}$ summands.

    Solution. The maps $\pi_n(F) \to \pi_n(E)$ in the long exact sequence of homotopy groups for a Serre fibration are induced by the inclusion $F \to E$, so if this is homotopic to a constant map, then the induced map is 0. Hence for all $n > 0$, we have short exact sequences

$$0 \longrightarrow \pi_n(E) \longrightarrow \pi_n(B) \longrightarrow \pi_{n-1}(F) \longrightarrow 0\,.$$

    Since $E \to B$ has the homotopy lifting property with respect to all disks, we can find a section $\pi_n(B) \to \pi_n(E)$ for the induced map $\pi_n(B) \to \pi_n(E)$, which means that the above short exact sequence splits. □

32. **Exercise.** Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n-1$ and $m = 2n-1$.

    Solution. We have the relations $n \leq m$ and $k \leq m$ and $k+n = m$. If $k = m$, then $n = 0$, and S^0 is not connected, so this contradicts that $S^m \to S^n$ is surjective. So $k < m$, and hence $S^k \to S^m$ is homotopic to a constant map. From (Ex. 4.2.31), we have $\pi_i(S^n) \cong \pi_i(S^m) \oplus \pi_{i-1}(S^k)$ for all $i > 0$. This shows that $k > 0$, so $m > n$. In particular, considering values of $i = 1, \ldots, n$, we see that $\pi_i(S^k) = 0$ if $i < n-1$ and $\pi_{n-1}(S^k) = \mathbf{Z}$, so $k = n-1$. Hence $m = 2n-1$. □

33. **Exercise.** Show that if there were fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ for all n, then the groups $\pi_i(S^n)$ would be finitely generated free Abelian groups computable by induction, and nonzero for $i \geq n \geq 2$.

    Solution. Assuming that fiber bundles $S^{n-1} \to S^{2n-1} \to S^n$ exist for all n, we can compute $\pi_i(S^n)$ by double induction on $i-n$ and n. Of course, if $i - n < 0$, then $\pi_i(S^n) = 0$, and if $i - n = 0$, then $\pi_n(S^n) = \mathbf{Z}$. Using (Ex. 4.2.31), we have $\pi_i(S^n) \cong \pi_i(S^{2n-1}) \oplus \pi_{i-1}(S^{n-1})$ for all n and $i > 0$. So if $\pi_j(S^m)$ is a finitely generated free Abelian group for all $j - m < i - n$ and $m < n$, then this shows that $\pi_i(S^n)$ is also a finitely generated free Abelian group. □

34. **Exercise.** Let $p\colon S^3 \to S^2$ be the Hopf bundle and let $q\colon T^3 \to S^3$ be the quotient map collapsing the complement of a ball in the 3-dimensional torus $T^3 = S^1 \times S^1 \times S^1$ to a point. Show that $pq\colon T^3 \to S^2$ induces the trivial map on π_* and $\widetilde{H}_*$, but is not homotopic to a constant map.

    Solution. The only nontrivial homotopy group of T^3 is $\pi_3(T^3) \cong \mathbf{Z}^3$. The map $q_*\colon \pi_3(T^3) \to \pi_3(S^3)$ is zero because any loop that goes around one of the factors of S^1 in T^3 can be homotoped to miss the ball that is used in the quotient map $T^3 \to S^3$. Hence pq induces the trivial map on all homotopy groups. Similarly, the only nontrivial reduced homology group of S^2 is $\widetilde{H}_2(S^2) = \mathbf{Z}$. The map that pq induces on homology factors as $\widetilde{H}_2(T^3) \to \widetilde{H}_2(S^3) \to \widetilde{H}_2(S^2)$, but since $\widetilde{H}_2(S^3) = 0$, this composition is 0. Hence pq also induces trivial maps on reduced homology.

    Note however, that a homotopy from pq to a constant map would give a homotopy from p to a constant map, so pq is not homotopic to a constant map. □

4.3. Connections with Cohomology

1. **Exercise.** Show there is a map $\mathbf{RP}^\infty \to \mathbf{CP}^\infty = K(\mathbf{Z}, 2)$ which induces the trivial map on $\widetilde{H}_*(-;\mathbf{Z})$ but a nontrivial map on $\widetilde{H}^*(-;\mathbf{Z})$. How is this consistent with the universal coefficient theorem?

    Solution. Note that $\widetilde{H}_n(\mathbf{RP}^\infty;\mathbf{Z})$ is $\mathbf{Z}/2$ for n odd and 0 otherwise, and that $\widetilde{H}_n(\mathbf{CP}^\infty;\mathbf{Z})$ is $\mathbf{Z}$ for $n > 0$ even and 0 otherwise, both of which can be seen by cellular homology and the fact that $\mathbf{RP}^\infty$ can be taken to have one cell for each dimension (the attaching map has degree 2), and $\mathbf{CP}^\infty$ has one cell for each even dimension. Hence every map $\mathbf{RP}^\infty \to \mathbf{CP}^\infty$ induces a trivial map on reduced homology. The homotopy classes of maps $\mathbf{RP}^\infty \to \mathbf{CP}^\infty$ are in bijection with cohomology classes $\beta \in H^2(\mathbf{RP}^\infty;\mathbf{Z})$; in particular, there is a distinguished class $\alpha \in H^2(\mathbf{CP}^\infty;\mathbf{Z})$ such that $f\colon \mathbf{RP}^\infty \to \mathbf{CP}^\infty$ gives $f^*(\alpha) = \beta$ (Theorem 4.57). Since the cohomology group $H^2(\mathbf{RP}^\infty;\mathbf{Z})$ is nonzero, we can find thus find a map $\mathbf{RP}^\infty \to \mathbf{CP}^\infty$ which is nontrivial on cohomology groups.

    This is consistent with the universal coefficient theorem because $\mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}) = 0$. □

2. **Exercise.** Show that the group structure on S^1 coming from multiplication in $\mathbf{C}$ induces a group structure on $\langle X, S^1\rangle$ such that the bijection $\langle X, S^1\rangle \to H^1(X;\mathbf{Z})$ of Theorem 4.57 is an isomorphism.

Solution. Given two maps $f, g\colon X \to \mathrm{S}^1$, let their sum $f+g$ be defined by $(f+g)(x) = f(x)g(x)$ where the multiplication is in $\mathbf{C}$. Suppose that f is homotopic to f' via H_1 and g is homotopic to g' via H_2. Then $f+g$ is homotopic to $f'+g'$ via the map $H_1 + H_2\colon X \times [0,1] \to \mathrm{S}^1$ which is given by $(x,t) \mapsto H_1(x,t)H_2(x,t)$. Hence this is a well-defined (Abelian) group structure on $\langle X, \mathrm{S}^1\rangle$.

Let $T\colon \langle X, \mathrm{S}^1\rangle \to \mathrm{H}^1(X;\mathbf{Z})$ be the bijection of Theorem 4.57. Then there is a distinguished class $\alpha \in \mathrm{H}^1(\mathrm{S}^1;\mathbf{Z})$ such that $T([f]) = f^*(\alpha)$. It is clear from the definition of pullback that $f^*(\alpha) + g^*(\alpha) = (f+g)^*(\alpha)$, so T is a group isomorphism. □

4. **Exercise.** Given Abelian groups G and H and CW complexes $K(G,n)$ and $K(H,n)$, show that the map $\langle K(G,n), K(H,n)\rangle \to \mathrm{Hom}(G,H)$ sending a homotopy class $[f]$ to the induced homomorphism $f_*\colon \pi_n(K(G,n)) \to \pi_n(K(H,n))$ is a bijection.

   Solution. Surjectivity of the map follows from Lemma 4.31. Now suppose we have two maps $f, g\colon K(G,n) \to K(H,n)$ such that $f_* = g_*$. In other words, for every basepoint-preserving map $\varphi\colon \mathrm{S}^n \to K(G,n)$, there is a homotopy $H_\varphi\colon \mathrm{S}^n \times [0,1] \to K(H,n)$ from $f \circ \varphi$ to $g \circ \varphi$. Letting φ vary over the characteristic maps of the n-cells of $K(G,n)$ shows that f is homotopic to g. More precisely, these define homotopies on the n-skeleton on $K(G,n)$, and the homotopy on the rest of the cells can be constructed using Lemma 4.7. □

5. **Exercise.** Show that $[X, \mathrm{S}^n] \cong \mathrm{H}^n(X;\mathbf{Z})$ if X is an n-dimensional CW complex.

   Solution. We can build a $K(\mathbf{Z},n)$ from S^n by attaching cells of dimension $\geq n+2$. The inclusion $\mathrm{S}^n \hookrightarrow K(\mathbf{Z},n)$ induces a map $\varphi\colon [X,\mathrm{S}^n] \to [X, K(\mathbf{Z},n)]$. If $\varphi(f) = \varphi(g)$, then there is a homotopy $H\colon X \times [0,1] \to K(\mathbf{Z},n)$ between f and g. By cellular approximation, this can be made to have image inside of the $(n+1)$-skeleton of $K(\mathbf{Z},n)$, which is equal to S^n, and hence $f = g$, so φ is injective. Surjectivity of φ also follows from cellular approximation since X is n-dimensional. Thus, $[X,\mathrm{S}^n] \cong [X, K(\mathbf{Z},n)] \cong \mathrm{H}^n(X;\mathbf{Z})$. □

6. **Exercise.** Use Exercise 4 to construct a multiplication map $\mu\colon K(G,n) \times K(G,n) \to K(G,n)$ for any Abelian group G, making a CW complex $K(G,n)$ into an H-space whose multiplication is commutative and associative up to homotopy and has a homotopy inverse. Show also that the H-space multiplication μ is unique up to homotopy.

   Solution. First note that $K(G,n)\times K(G,n) \simeq K(G\times G, n)$. By (Ex. 4.3.4), there is a bijection $\langle K(G\times G,n), K(G,n)\rangle \cong \mathrm{Hom}(G\times G, G)$. Let $\mu\colon K(G,n)\times K(G,n) \to K(G,n)$ be a map (well-defined only up to homotopy) corresponding to the map $G \times G \to G$ given by $(g,g') \mapsto g + g'$. From the naturality of this isomorphism, it follows that μ is associative and commutative up to homotopy. There is a homotopy inverse given by letting $i\colon K(G,n) \to K(G,n)$ correspond to the homomorphism $G \to G$ given by $g \mapsto -g$. If μ' is another such H-space multiplication on $K(G,n)$, then it must correspond to the addition map $G \times G \to G$ by (Ex. 4.1.3). □

7. **Exercise.** Using an H-space multiplication μ on $K(G,n)$, define an addition in $\langle X, K(G,n)\rangle$ by $[f]+[g] = [\mu(f,g)]$ and show that under the bijection $\mathrm{H}^n(X;G) \cong \langle X, K(G,n)\rangle$ this addition corresponds to the usual addition in cohomology.

   Solution. This follows as in (Ex. 4.3.2). □

8. **Exercise.** Show that a map $p\colon E \to B$ is a fibration if and only if the map $\pi\colon E^I \to E_p$, $\pi(\gamma) = (\gamma(0), p\gamma)$, has a section.

Solution. First suppose that π has a section $s\colon E_p \to E^I$. Let $g_t\colon X \to B$ be a homotopy, and $\widetilde{g}_0\colon X \to E$ be a lift of g_0. Then define $\widetilde{\gamma}_t\colon X \to E$ by $x \mapsto (s(\widetilde{\gamma}_0(x), \gamma_t))(t)$. Since s is a section of π, we have $p \circ s(\widetilde{\gamma}_0(x), \gamma_t) = \gamma_0(x)$, so $\widetilde{\gamma}_t$ is a lift of γ_t, and hence p is a fibration.

Now suppose that p is a fibration. Given $(e, \gamma) \in E_p$, i.e., $e \in E$ and $\gamma\colon I \to B$ with $\gamma(0) = p(e)$, define $s(e, \gamma)$ as follows. We have a map $* \to B$ given by $* \mapsto p(e)$, and a homotopy $h_t\colon * \times [0,1] \to B$ given by γ. The point e provides a lift $\widetilde{h}_0$ of h_0, and the (unique) lift $\widetilde{h}_t$ of h_t is an element of E^I which we define to be $s(e, \gamma)$. It follows immediately that s is a section of π. □

9. **Exercise.** Show that a linear projection of a 2-simplex onto one of its edges is a fibration but not a fiber bundle.

Solution. Let Δ be a 2-simplex, and let I be one of its edges, with $p\colon \Delta \to I$ linear projection. The map $\pi\colon \Delta^I \to \Delta_p$ given by $\gamma \mapsto (\gamma(0), p\gamma)$ has a section $s\colon \Delta_p \to \Delta^I$ where $s(x, \gamma)\colon I \to \Delta$ is the map $s(x, \gamma)(t) = \gamma(t)$. So by (Ex. 4.3.8), p is a fibration. However, it is not a fiber bundle because the fibers over the vertices of the edge are points, while the other fibers are line segments, and hence not homeomorphic. □

11. **Exercise.** For a space B, let $\mathcal{F}(B)$ be the set of fiber homotopy equivalence classes of fibrations $E \to B$. Show that a map $f\colon B_1 \to B_2$ induces $f^*\colon \mathcal{F}(B_2) \to \mathcal{F}(B_1)$ depending only on the homotopy class of f, with f^* a bijection if f is a homotopy equivalence.

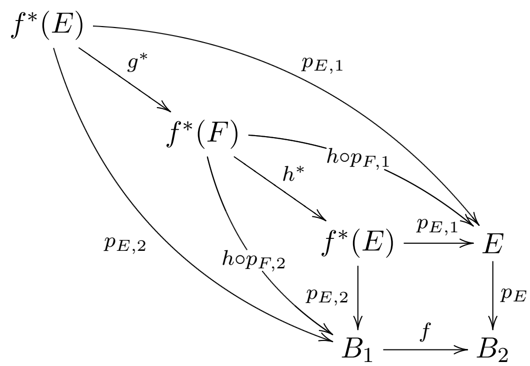
Solution. Given a fibration $p\colon E \to B_2$, let $f^*(E)$ be the pullback $f^*(E) \to B_1$ along f, i.e.,

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

is a pullback diagram. We have to show that if $p_E\colon E \to B_2$ and $p_F\colon F \to B_2$ are fiber homotopy equivalent fibrations over B_2, then so are $f^*(E)$ and $f^*(F)$. Let $g\colon E \to F$ and $h\colon F \to E$ be fiber-preserving maps such that gh and hg are homotopic to the identity through fiber-preserving maps. Let $p_{F,1}\colon f^*(F) \to F$, $p_{F,2}\colon f^*(F) \to B_1$, $p_{E,1}\colon f^*(E) \to E$, and $p_{E,2}\colon f^*(E) \to B_1$ be the respective projection maps. The composition $f^*(F) \to F \to E$ gives a commutative diagram

$$\begin{array}{ccc} f^*(F) & \longrightarrow & E \\ {\scriptstyle p_{F,2}}\downarrow & & \downarrow{\scriptstyle p_E} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

and hence by the universal property of pullback, we have an induced map $h^*\colon f^*(F) \to f^*(E)$. Similarly, we get an induced map $g^*\colon f^*(E) \to f^*(F)$. Since these maps fit into diagrams consisting of fiber-preserving maps, they are also fiber-preserving. The following diagram



commutes up to homotopy, so by uniqueness of induced maps, $h^* \circ g^*$ is homotopic to the identity of $f^*(E)$ through fiber-preserving maps. Similarly, $g^* \circ h^*$ is homotopic to the identity of $f^*(F)$ through fiber-preserving maps. So we have a well-defined function $f^*\colon \mathcal{F}(B_2) \to \mathcal{F}(B_1)$. That f^* only depends on the homotopy class of f is the content of Proposition 4.62.

Finally, it is clear that if $B_1 = B_2$ and f is the identity map, then f^* is also the identity map. Also, it is clear that $(f \circ g)^* = g^* \circ f^*$ from the associativity of pullback. So if $f\colon B_1 \to B_2$ is a homotopy equivalence with homotopy inverse $g\colon B_2 \to B_1$, then $f^* \circ g^*$ and $g^* \circ f^*$ are the identity maps on $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$, respectively. Hence f^* is a bijection. □

12. **Exercise.** Show that for homotopic maps $f, g\colon A \to B$ the fibrations $E_f \to B$ and $E_g \to B$ are fiber homotopy equivalent.

    Solution. Let H be a homotopy from f to g, and let $\overline{H}$ be the reverse homotopy from g to f. Define a map $E_f \to E_g$ by $(a, \gamma) \mapsto (a, \overline{H}(a) \cdot \gamma)$ where the $\cdot$ denotes the path which travels $\overline{H}(a)$ first at double speed, and then γ at double speed. Similarly, we can define a map $E_g \to E_f$ by $(a, \gamma) \mapsto (a, H(a) \cdot \gamma)$. Then it is clear that these maps are fiber-preserving and that their compositions are homotopic to identity maps through homotopy-preserving maps, so E_f and E_g are fiber homotopy equivalent. □

13. **Exercise.** Given a map $f\colon A \to B$ and a homotopy equivalence $g\colon C \to A$, show that the fibrations $E_f \to B$ and $E_{fg} \to B$ are fiber homotopy equivalent.

    Solution. Since A is homotopy equivalent to the mapping cylinder M_g, we may assume that $g\colon C \to A$ is a deformation retract by Corollary 0.21. In this case, E_{fg} is a deformation retract of E_f because $f(g(C))$ is a deformation retract of $f(A)$. □

14. **Exercise.** For a space B, let $\mathcal{M}(B)$ denote the set of equivalence classes of maps $f\colon A \to B$ where $f_1\colon A_1 \to B$ is equivalent to $f_2\colon A_2 \to B$ if there exists a homotopy equivalence $g\colon A_1 \to A_2$ such that $f_1 \simeq f_2 g$. Show the natural map $\mathcal{F}(B) \to \mathcal{M}(B)$ is a bijection.

    Solution. A fibration $E \to B$ is an element of $\mathcal{M}(B)$, and two fiber homotopy equivalent fibrations are equivalent as elements of $\mathcal{M}(B)$. So we have a natural map $\mathcal{F}(B) \to \mathcal{M}(B)$. This map is surjective because any map $f\colon A \to B$ is equivalent to the fibration $E_f \to B$ since the natural inclusion $A \hookrightarrow E_f$ is a homotopy equivalence. Injectivity follows because if two fibrations $p_1\colon E_1 \to B$ and $p_2\colon E_2 \to B$ are homotopy equivalent via $g\colon E_1 \to E_2$ such that $p_1 \simeq p_2 g$, then $E_{p_1} \to B$ and $E_{p_2} \to B$ are fiber homotopic fibrations by (Ex. 4.3.13). Hence $E_1 \to B$ and $E_2 \to B$ are fiber homotopic equivalent (Proposition 4.65). □

15. **Exercise.** If the fibration $p\colon E \to B$ is a homotopy equivalence, show that p is a fiber homotopy equivalence of E with the trivial fibration $1\colon B \to B$.

Solution. In this case, p is a fiber-preserving map, and a homotopy inverse q of p can be chosen to be fiber-preserving by (Ex. 4.3.14). □

17. **Exercise.** Show that ΩX is an H-space with multiplication the composition of loops.

Solution. The identity is the constant loop of ΩX. Since composition of loops is associative up to homotopy, and composition of a loop with the constant loop is homotopic to itself, this gives an H-space structure on ΩX. □

18. **Exercise.** Show that a fibration sequence $\cdots \to \Omega B \to F \to E \to B$ induces a long exact sequence $\cdots \to \langle X, \Omega B\rangle \to \langle X, F\rangle \to \langle X, E\rangle \to \langle X, B\rangle$, with groups and group homomorphisms except for the last three terms, Abelian groups except for the last six terms.

Solution. We give $\langle X, \Omega^n K\rangle$ the structure of a group as in (Ex. 4.3.2), where K is any space and $n > 0$. If $n > 1$, then $\Omega^{n-1}K$ is an H-space by (Ex. 4.3.17), so composition of loops in $\Omega^n K$ is commutative up to homotopy (Ex. 3.C.5), and hence $\langle X, \Omega^n K\rangle$ has the structure of an Abelian group.

We first show that if $F \to E \to B$ is a fibration, then $\langle X, F\rangle \to \langle X, E\rangle \to \langle X, B\rangle$ is exact. It is obvious that the composition of the two maps is zero. Now let $f \in \langle X, E\rangle$ be such that pf is homotopic to a constant map where $p\colon E \to B$ is the projection. Then we can use the homotopy lifting property to homotope f to a map that lives inside of the fiber of the basepoint of E, and hence the sequence is exact.

A map of H-spaces $E \to B$ preserving the multiplication induces a group homomorphism $\langle X, E\rangle \to \langle X, B\rangle$. So we need to show that given $E \to B$, the induced map $\Omega E \to \Omega B$ preserves the multiplication. Since the multiplication is induced by composition of loops, this follows. □

19. **Exercise.** Given a fibration $F \to E \xrightarrow{p} B$, define a natural action of ΩB on the homotopy fiber F_p and use this to show that exactness at $\langle X, F\rangle$ in the long exact sequence in the preceding problem can be improved to the statement that two elements of $\langle X, F\rangle$ have the same image in $\langle X, E\rangle$ if and only if they are in the same orbit of the induced action of $\langle X, \Omega B\rangle$ on $\langle X, F\rangle$.

Solution. Pick $\gamma \in \Omega B$ and $(e, \eta) \in F_p$, i.e., $\eta\colon [0,1] \to B$ such that $\eta(0) = p(e)$ and $\eta(1) = b_0$, where $b_0 \in B$ is the basepoint. Note that $\overline{\eta} \cdot \gamma \cdot \eta$ is a homotopy from $p(e)$ to itself, and e is a lift of $p(e)$, so by the homotopy lifting property, we get a homotopy $\widetilde{\eta}\colon [0,1] \to E$ lifting $\overline{\eta} \cdot \gamma \cdot \eta$. Define $\gamma \cdot (e, \eta) = (\widetilde{\eta}(1), \eta)$. The endpoint is independent of the lifting chosen. This defines an action of ΩB on F_p since $(\gamma \cdot \gamma') \cdot (e, \eta) = \gamma \cdot (\gamma' \cdot (e, \eta))$ by definition.

Two elements $f, g \in \langle X, F\rangle$ have the same image in $\langle X, E\rangle$ if and only if there is a homotopy $H\colon X \times [0,1] \to E$ through basepoint-preserving maps from f to g. Such a homotopy is the same as the existence of an action of an element of ΩB taking $f(x)$ to $g(x)$ for all $x \in X$, and hence f and g are in the same orbit of ΩB. □

20. **Exercise.** Show that by applying the loopspace functor to a Postnikov tower for X one obtains a Postnikov tower of principal fibrations for ΩX.

Solution. Let $\cdots \to X_2 \to X_1$ be a Postnikov tower for X. Applying the loopspace functor gives a Postnikov tower $\cdots \to \Omega X_2 \to \Omega X_1$ for ΩX. By the discussion on p. 409, $\Omega X_{n+1} \to \Omega X_n \to \Omega X_{n-1}$ is a principal fibration for all $n > 1$. □

23. **Exercise.** Prove the following uniqueness result for the Quillen plus construction: Given a connected CW complex X, if there is an Abelian CW complex Y and a map $X \to Y$ inducing an isomorphism $\mathrm{H}_*(X; \mathbf{Z}) \cong \mathrm{H}_*(Y; \mathbf{Z})$, then such a Y is unique up to homotopy equivalence.

Solution. Let $X \to Y'$ be another map which induces isomorphisms on homology such that Y' is an Abelian CW complex. Let W be the mapping cylinder of $X \to Y$. Then $\mathrm{H}^n(W) \cong \mathrm{H}^n(X)$ via the inclusion $X \hookrightarrow W$ by hypothesis that $X \to Y$ induces isomorphisms on homology. Hence $\mathrm{H}^{n+1}(W, X; \pi_n(Y')) = 0$ for all n by the long exact sequence on cohomology for the pair (W, X). By Corollary 4.73, there is a lift $W \to Y'$ of the map $X \to Y'$. In particular, this means that we have a map $Y \to Y'$ commuting with the maps $X \to Y$ and $X \to Y'$. So this map must induced isomorphisms on homology, and hence is a homotopy equivalence (Proposition 4.74). $\square$