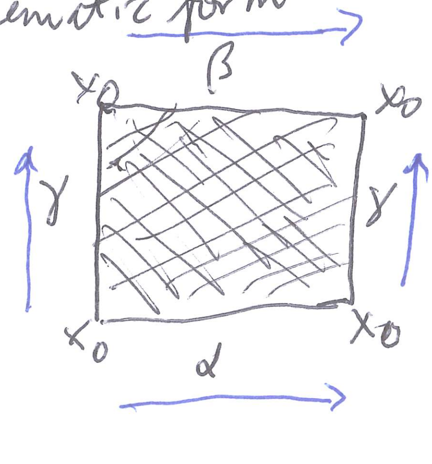
A deformation retract construction

The proof of the result about the map

$$\pi_1(X, x_0) \xrightarrow[\text{basepts}]{\text{forget}} [S^1, X]$$

in the lectures differed from that in the notes at one step. We shall give a more detailed version of the alternative argument here. $\boxed{X \text{ is arcwise conn.}}$

RECALL We wanted to show that if $[\alpha]$ and $[\beta]$ in $\pi_1(X, x_0)$ go to the same element of $[S^1, X]$, then $[\alpha]$ and $[\beta]$ are conjugate: There $\exists$ some $[\gamma]$ such that $[\beta] = [\gamma]^{-1}[\alpha][\gamma]$. Viewing α and β as closed curves, ~~∃~~ a free homotopy $\alpha \simeq \beta$ has the schematic form

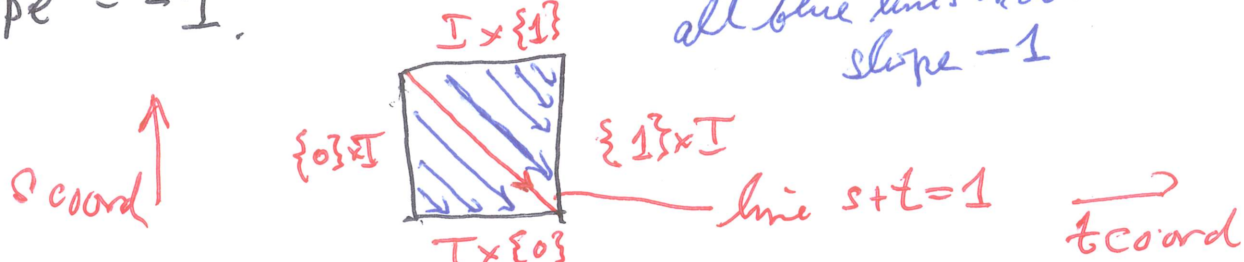


on $I \times I$

If γ is constant, this homotopy would be base pt. preserving, but we don't know if this is true!

The idea in the lectures was to show

$[\alpha][\gamma] = [\gamma][\beta]$ from this by constructing an ~~free~~ end point preserving homotopy $\alpha + \gamma \underset{*}{\simeq} \gamma + \beta$, where the homotopy moves points along lines of slope $= -1$.



In fact we can say more!

CLAIM There is a retraction $I \times I \to I \times \{0\} \cup \{1\} \times I$ which is the identity on $I \times \{0\} \cup \{1\} \times I$ and maps $\{0\} \times I$ and $I \times \{1\}$ linearly to $I \times \{0\}$ and $\{1\} \times I$ respectively. If $j : I \times \{0\} \cup \{1\} \times I \to I \times I$ is inclusion, then $j \circ r$ is homotopic to the identity, where the homotopy moves along lines of slope -1 and is fixed on $I \times \{0\} \cup \{1\} \times I$.

Verification

The point (s,t) lies on the line $x+y=s+t$, so we need to find where this line meets $I\times\{0\}\cup\{1\}\times I$, ~~and then~~ which gives us $r(s,t)$, and then we can take a straight line homotopy. The definition of $r(s,t)$ has two cases, one where the line with slope -1 meets the bottom edge and one where the line meets the right edge (of the square). The drawing suggests these subsets meet on the line $x+y=1$.

Formally, let

$$r(s,t)=\begin{cases}(1,\, s+t-1) & \text{if } s+t\geq 1 \quad (\text{hence } s+t-1\in[0,1]) \\ (s+t,0) & \text{if } s+t\leq 1\end{cases}$$

Check that these maps' images are in $\{1\}\times I$ and $I\times\{0\}$ respectively when $(s,t)\in I\times I$, and on the overlap set where $s+t=1$ both formulas yield the point $(1,0)$, so that $r(s,t)$ is well-defined (and continuous since $\left\{\begin{matrix} s+t\geq 1 \\ s+t\leq 1\end{matrix}\right\}$ are closed sets).

Now take the homotopy

$$H(s,t;u) \quad (I \times I) \times I \longrightarrow I \times I$$

"time"

defined by $(1-u) \cdot r(s,t) + u \cdot (s,t)$.

Finally, one must check that r and H have all the properties claimed on page 2 ∎