Comment (RS)

These problems are meant to challenge students, so it is very good if the solutions are understood but nothing at this level would be on a qualifying exam.

SOLUTION TO EXERCISE 3 (Sketch)

<u>Reduction</u> If this is true for bounded open sets it is true for all open sets.

To see this, note that if $z \in H_q(U)$ then $z \in \text{Image } H_q(K)$ for some compact $K \subseteq U$.

Let $z' \in H_q(K)$ map to z

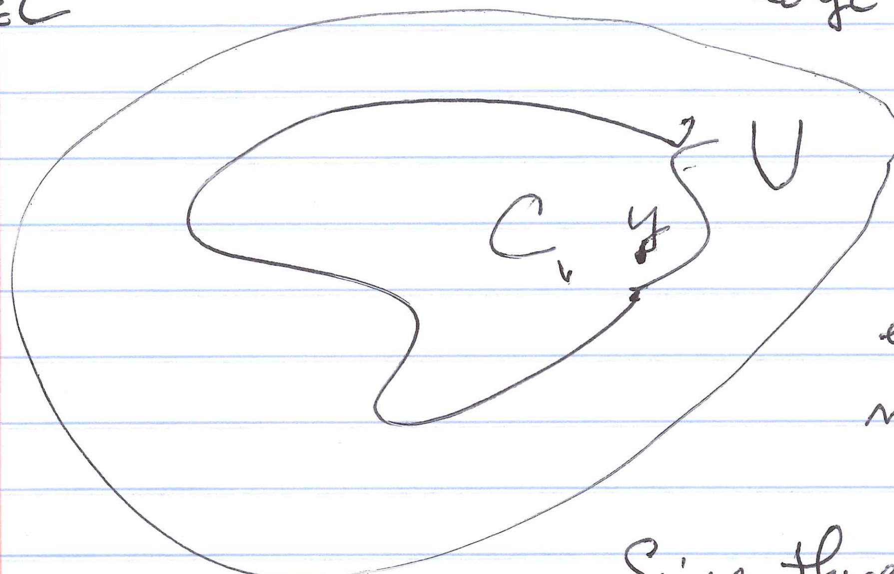
If V is open such that $K \subseteq V \subseteq \bar{V}$ and $\bar{V}$ is compact, then $z' \to 0$ in $H_q(V)$, so it must go to zero in $H_q(U)$.

<u>Assume now that U is bounded.</u>

Define L_k as in the statement of the exercise. Then $H_q(L_k) = 0$ for $q > n$ because L_k is an n-dim simplicial complex. If every compact set $C \subseteq U$ is contained in some L_k, then every class in $H_q(U)$ is in the image of some $H_q(L_k)$, and since the latter are zero if $q > n$ it follows that $H_q(U) = 0$ if $q > n$.

Let $\delta =$ distance$(C, \mathbb{R}^n - U)$; this is positive. Therefore, if we choose k so large that the diameter of a hypercube with edge 2^{-k} is less than, say, $\delta/2$,

Let $y \in C$



then every hypercube with edges of this length must be contained in U.

Since there is some hypercube of this length with vertices of the prescribed form which contains $y \in C$, it follows that $y \in L_k$. Since y was arbitrary we have $C \subseteq L_k$ and we are done. ∎

SOLUTION TO EXERCISES 4 (Sketch)

Use the notation in the statement of the exercise, and let $p \in U_+ \cap U_-$. Then $X \times \{p\}$ is a retract of $X \times S^k$ so

$$H_*(X \times S^k) \cong H_*(X \times \{p\}) \oplus H_*(X \times S^k, X \times \{p\}).$$

It suffices to show that the second summand is isomorphic to $H_{*-k}(X)$.

We have the following "diagram" in which the horizontal maps are the slice inclusions $X \cong X \times \{p\} \subseteq X \times [?]$:

$$(X \times \{p\}; X \times \{p\}, X \times \{p\}) \subseteq (X \times S^k, X \times U_+, X \times U_-)$$

This yields a map of long exact MV sequences:

maps induced by slice inclusion

$$\begin{tikzcd}
H_{q+1}(X) \arrow[r, "\Delta_0"] \arrow[d, "s_3"] & H_q(X) \arrow[r, "{(id,-id)}"] \arrow[d, "s_2"] & H_q(X) \oplus H_q(X) \arrow[r, "id+id"] \arrow[d, "s_1 \oplus s_0"] & H_q(X) \arrow[r, "\Delta_0"] \arrow[d, "s_3"] & \cdots \\
H_{q+1}(X \times S^k) \arrow[r, "\Delta"] & H_q(X \times U_+ \cap U_-) \arrow[r] & H_q(X \times U_+) \oplus H_q(X \times U_-) \arrow[r] & H_q(X \times S^k) \arrow[r, "\Delta"] & \cdots
\end{tikzcd}$$

Each S_j comes from a retract, so what we want are the quotients. Also, clearly $\Delta_0 = 0$. One can check directly that the complement any summands to the S_j's also form an exact sequence

$$\to C_{q+1}(X \times S^k) \to C_q(X \times (U_+ \cap U_-)) \to \begin{matrix} C_q \\ \oplus \\ C_q \end{matrix} \to \cdots$$

Now S_1 & S_0 are isos, so $C_q \oplus C_q = 0$, and hence

$$C_{q+1}(X \times S^k) \cong C_q(X \times (U_+ \cap U_-)).$$

Since $S^{k-1} \subseteq U_+ \cap U_-$ is a homotopy equivalence, we have $C_q(X \times (U_+ \cap U_-)) \cong C_q(X \times S^{k-1})$ and <u>by induction</u> the latter is isomorphic to $H_{q-(k-1)}(X)$. Therefore

$$C_n(X \times S^k) \cong C_{n-1}(X \times S^{k-1}) \cong H_{(n-1)-(k-1)}(X) = H_{n-k}(X). \blacksquare$$