

CLASSIFYING n -SHEETED COVERINGS

$p: E \rightarrow B$ covering space projection
 E, B satisfy default hypotheses plus
local simple connectivity. Also E, B
connected.

Regular coverings Classify all ^{regular based*} coverings
with $\Gamma(p) = G$, where $|G| = n$.
 \uparrow
group of covering transformations

Solution The basic results state such
coverings are classified by normal subgps
 $N \triangleleft \pi_1(B, b_0)$ s.t. $G \cong \pi_1(B, b_0) / N$.

Claim These are in 1-1 correspondence
with surjective homomorphisms

$$\pi_1(B, b_0) \xrightarrow{h} G, \text{ modulo } h \sim h' \Leftrightarrow$$

$$h' = \varphi \circ h \text{ for some automorphism } \varphi \text{ of } G.$$

*based: p is basepoint preserving

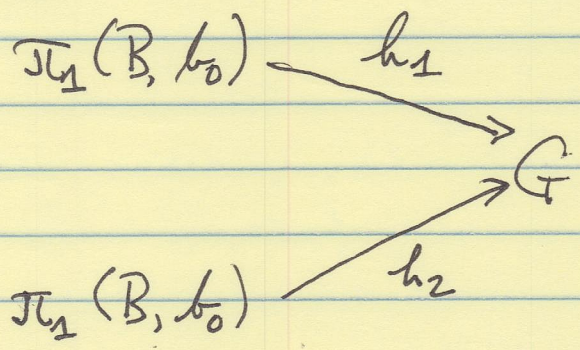
Proof of claim. Let \mathcal{S} be the set of normal subgroups as above, and let

$\mathcal{H} =$ onto homomorphisms $\pi_1(B, b_0) \rightarrow G$.

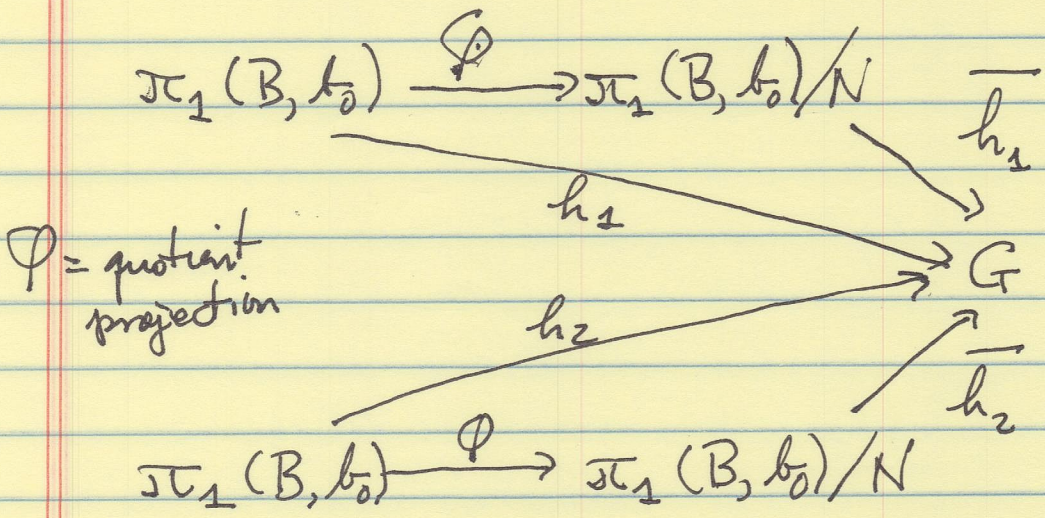
Then there is an onto map $\mathcal{H} \rightarrow \mathcal{S}$ sending an onto homomorphism h to $\text{Kernel } h$.

Suppose now that $\text{Kernel } h_1 = \text{Kernel } h_2 = N$

We then have a commutative diagram as follows



We can rewrite this as follows



Both $\overline{h_1}$ and $\overline{h_2}$ are then isomorphisms, so

$$\bar{h}_1^{-1} \circ h_1 = \varphi = \bar{h}_2^{-1} \circ h_2 \text{ and hence}$$

$$h_1 = \bar{h}_1 \circ \bar{h}_2^{-1} \circ h_2 =$$

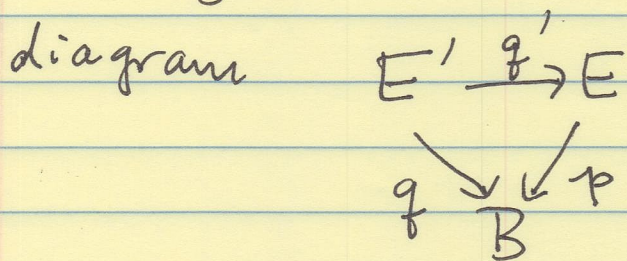
$$\Phi \circ h_2, \text{ where } \Phi \text{ is}$$

an ^{auto} ~~isomorphism~~ isomorphism of G . Conversely, if

$h_1 = \Phi \circ h_2$ for some automorphism Φ of G , then $\text{Kernel } h_1 = \text{Kernel } h_2$. \blacksquare

Non regular (or not necessarily regular) n -sheeted coverings.

Exercise 01. shows that if $E \rightarrow B$ is n -sheeted, then there is a finite regular covering $E' \rightarrow E$ and a commutative



such that

$$\Gamma(q) \subseteq \sum_{\text{in}} \text{symmetric group}$$

In fact, $\Gamma(q)$ permutes $\{1, \dots, n\}$

transitively; i.e. $\Gamma(q)$ is not conjugate to

a subgroup of the form $\Sigma_k \times \Sigma_{m-k} \subseteq \Sigma_m$.

Furthermore, we have $|\Gamma(q)/\Gamma(q) \cap \Sigma_{m-1}| = m$,

and $E'/\Gamma(q) \cap \Sigma_{m-1} \cong E$.

Non regular coverings correspond to examples where $\Gamma(q) \cap \Sigma_{m-1}$ is not normal in $\Gamma(q)$.

Application to 3-sheeted coverings:

$\Gamma(q) \subseteq \Sigma_3$ and $3 \mid |\Gamma(q)| \Rightarrow$

either $\Gamma(q) \cong \mathbb{Z}_3$ (regular case)

or $\Gamma(q) \cong \Sigma_3$ (irregular case) with

$\Gamma(q) \cap \Sigma_2 = \Sigma_2$.

Hence irregular 3-sheeted coverings \Leftrightarrow
subgroups N such that $N \triangleleft \pi_1(B, b_0)$

and $\pi_1(B, b_0)/N \cong \Sigma_3$. \blacksquare

* \Downarrow $N^* \subseteq \pi_1(B, b_0)$ is the inverse image of Σ_2 ,
then $\pi_1(E, e_0) \cong N^*$.