CLASSIFYING n-SHEETED COVERINGS

$p: E \to B$ covering space projection

E + B satisfy default hypotheses plus local simple connectivity. Also E, B connected.

<u>Regular coverings</u> Classify all regular based* coverings with $\Gamma(p) = G$, where $|G| = n$.

group of covering transformations

<u>Solution</u> The basic results state such coverings are classified by normal subgps $N \triangleleft \pi_1(B, b_0)$ s.t. $G \cong \pi_1(B, b_0)/N$.

<u>Claim</u> These are in 1-1 correspondence with surjective homomorphisms $\pi_1(B, b_0) \xrightarrow{h} G$, modulo $h \sim h' \iff$ $h' = \varphi \circ h$ for some automorphism φ of G.

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* based: p is basepoint preserving

<u>Proof of claim.</u> Let $\mathcal{S}$ be the set of normal subgroups as above, and let $\mathcal{H}$ = onto homomorphisms $\pi_1(B, b_0) \longrightarrow G$.
Then there is an onto map $\mathcal{H} \longrightarrow \mathcal{S}$ sending an onto homomorphism h to Kernel h.
Suppose now that Kernel h_1 = Kernel h_2 = N
We then have a commutative diagram as follows

$$\begin{tikzcd}
\pi_1(B, b_0) \arrow[dr, "h_1"] & \\
& G \\
\pi_1(B, b_0) \arrow[ur, "h_2"'] &
\end{tikzcd}$$

We can rewrite this as follows

$$\begin{tikzcd}
\pi_1(B, b_0) \arrow[r, "\Phi"] \arrow[drr, "h_1"'] & \pi_1(B, b_0)/N \arrow[dr, "\overline{h_1}"] & \\
& & G \\
\pi_1(B, b_0) \arrow[r, "\Phi"] \arrow[urr, "h_2"] & \pi_1(B, b_0)/N \arrow[ur, "\overline{h_2}"'] &
\end{tikzcd}$$

Φ = quotient projection

Both $\overline{h_1}$ and $\overline{h_2}$ are then isomorphisms, so

$\bar{h}_1^{-1} \circ h_1 = \varphi = \bar{h}_2^{-1} \circ h_2$ and hence

$$h_1 = \bar{h}_1 \circ \bar{h}_2^{-1} \circ h_2 =$$

$$\Phi \circ h_2$$, where Φ is

an ~~iso~~automorphism of G. Conversely, if $h_1 = \Phi \circ h_2$ for some automorphism Φ of G, then Kernel h_1 = Kernel h_2. ∎

<u>Non regular (or not necessarily regular) n-sheeted coverings.</u>

Exercise 01. shows that if $E \to B$ is n-sheeted, then there is a finite regular covering $E' \to E$ and a commutative diagram

$$\begin{tikzcd} E' \arrow[r, "q'"] \arrow[dr, "q"'] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

such that $\Gamma(q) \subseteq \Sigma_n$ symmetric group

In fact, $\Gamma(q)$ permutes $\{1, \dots, n\}$ transitively; i.e. $\Gamma(q)$ is not conjugate to a subgroup of the form $\Sigma_k \times \Sigma_{n-k} \subseteq \Sigma_n$. Furthermore, we have $|\Gamma(q)/\Gamma(q) \cap \Sigma_{n-1}| = n$, and $E'/\Gamma(q) \cap \Sigma_{n-1} \cong E$.

Non regular coverings correspond to examples where $\Gamma(q) \cap \Sigma_{n-1}$ is not normal in $\Gamma(q)$.

<u>Application to 3-sheeted coverings.</u>

$\Gamma(q) \subseteq \Sigma_3$ and $3 \mid |\Gamma(q)| \Rightarrow$ either $\Gamma(q) \cong \mathbb{Z}_3$ (regular case) or $\Gamma(q) \cong \Sigma_3$ (irregular case) with $\Gamma(q) \cap \Sigma_2 = \Sigma_2$.

Hence irregular 3-sheeted coverings $\leftrightarrow$ subgroups N such that $N \triangleleft \pi_1(B, b_0)$ and $\pi_1(B, b_0)/N \cong \Sigma_3$.* ■

* If $N^* \subseteq \pi_1(B, b_0)$ is the inverse image of Σ_2, then $\pi_1(E, e_0) \cong N^*$.