The Polish Circle and some of its unusual properties

We shall discuss some additional results on the space which is studied in Munkres, Section 61 (the *closed topologist's sine curve*, also known as the *Polish Circle*), and described more explicitly in `polishcircleA.pdf`.

None of this material will be used subsequently in topics to be covered on examinations, so it can be skipped without loss of continuity. However, it does illustrate some approaches and methods that appear frequently in more advanced topology courses, using only material within the setting of this course and its prerequisite. At one point in the discussion we shall need a result that might not received much attention in 205A; namely, the *Tietze Extension Theorem*, which states that if A is a closed subset of a metric space X and $f : A \to \mathbb{R}^n$ is continuous, then f extends to a continuous function on X (see Theorem 35.1 on pp. 219–222 of Munkres).

We begin with a couple of basic observations.

PROPOSITION 1. *Let P be the Polish circle as described as in the references cited above, and let $V_n \subset \mathbb{R}^2$ be the open rectangular region*

$$\left(0,\ \frac{2}{(4n+3)\pi}\right) \ \times\ \left(-\frac{3}{2}, \frac{3}{2}\right) \ .$$

Then $P - V_n$ is homeomorphic to a closed interval and hence is contractible.

Proof. We shall use the following decomposition of the Polish circle in `polishcircleA.pdf`:

(A) The graph A of $\sin(1/x)$ for $0 < x < 1$.

(B) The vertical line segment B joining $(1, \sin 1)$ to $(1, -2)$.

(C) The horizontal line segment C joining $(0, -2)$ to $(1, -2)$

(D) The vertical line segment D joining $(0, -2)$ to $(0, 1)$.

If we remove the points which lie in the rectangular region, we are left with the last three pieces plus the graph of $\sin(1/x)$ for

$$\frac{2}{(4n+3)\pi} \ \leq\ x\ \leq\ 1\ .$$

If we let A' denote the displayed set then A' is homeomorphic to a closed interval and the complement is homeomorphic to the union $A' \cup B \cup C \cup D$. We also have the intersection identities

$$A' \cap B \ = \ \{(1, \sin 1)\}\ , \quad B \cap C \ = \ \{(1, -2)\}\ , \quad C \cap D \ = \ \{(0, -2)\}\ .$$

It is now an elementary exercise to construct a homeomorphism from $[0, 4]$ to P such that $[0, 1]$ corresponds to A', $[1, 2]$ to B, $[2, 3]$ to C, and $[3, 4]$ to D.∎

Simple connectivity of the Polish Circle

PROPOSITION 2. *In the setting of the previous result, if K is a compact, connected and locally connected space, and $f : K \to P$ is continuous, then there is some $n > 0$ such that the image of f is contained in $P - V_n$.*

Sketch of proof. The main step involves the following standard observation: If $(x, 0) \in P$ such that $x \geq -1$, then for every sufficiently small open neighborhood W of $(x, 0)$ in P, the connected component of $(x, 0)$ in $P \cap W$ is contained in the y-axis. — This is the basic reason why the Polish circle is not locally connected.

Combining this with the local connectedness of K, we see that for every $y \in K$ there is an open neighborhood W_y and a positive integer $n(y)$ such that f maps W_y into $P - V_{n(y)}$. By compactness there is some $m > 0$ such that f maps K into $P - V_m$.■

COROLLARY 3. *If $x_0 \in P$, then $\pi_1(P, x_0)$ is trivial.*

Similar considerations show that if X is an arbitrary compact, arcwise connected and locally arcwise connected space, then every continuous map from X to P is homotopic to a constant map.

Proof. If γ is a closed curve in P, then by the previous proposition we know that the image of γ is contained in a set of the form $P - V_m$ for some m. However, these sets are contractible by the first proposition above, and therefore the class of γ in the fundamental group of P must be trivial.■

Noncontractibility of the Polish Circle

In contrast, it turns out that P is not a contractible space. This will be an immediate consequence of the following result, which reflects the geometric similarities between P and the standard circle S^1:

THEOREM 4. *If P is the Polish Circle, then there is a continuous map from P to S^1 which is not homotopic to a constant.*

COROLLARY 5. *The space P is not contractible.*

Proof. If P were contractible, then for every space Y, all continuous maps from P to Y would be homotopic to constant mappings.■

Proof of Theorem. We shall use the setting and terminology of `polishcircleA.pdf` freely in the discussion below. Define a mapping r_1 from B_1 to the boundary G of the square with vertices $(1, -1)$, $(0, -1)$, $(0, -2)$, and $(1, -2)$ such that G sends (x, y) to $\big(x, m(y)\big)$, where $m(y)$ is the lesser of y and -1. By construction, for every positive integer n the restriction of r_1 to the simple closed curve C_n is onto, it is 1–1 off the set $\{1\} \times [-1, \sin 1)$, and it is constant on that exceptional interval. If we compose G with standard homeomorphisms $S^1 \cong C_n$ and $G \cong S^1$, we obtain a mapping g_n from S^1 to itself. Furthermore, if $\varphi : [0, 1] \to S^1$ is the usual map $\varphi(t) = \exp(2\pi \mathbf{i} t)$, then $g_n \circ \varphi$ is a map such that g_n is onto and there are points $a_n < b_n$ in the open interval $(0, 1)$ such that $g_n \circ \varphi$

is 1–1 on both $[0, a_n]$ and $[b_n, 1]$, while it is constant on $[a_n, b_n]$. Furthermore, this function is 1–1 on the complement of $[a_n, b_n)$.

CLAIM. *The mapping g_n is homotopic to a homotopy equivalence.* — Let $t_n \in \mathbb{R}$ be such that $g_n(1) = p(t_n)$, where p denotes the standard covering map from $\mathbb{R}$ to S^1, and let γ_n denote the unique lifting of $g_n \circ \varphi$ such that $\gamma_n(0) = t_n$. Since γ_n is 1–1 on $[0, a_n]$ it follows that it is either strictly increasing or decreasing there. We shall only consider the case where γ_n is increasing. In the other case, the curve $-\gamma_n$ is an increasing lifting of the complex conjugate curve $\overline{g_n}$, and by the increasing case we know that this conjugate curve is homotopic to a homeomorphism; taking conjugates, we see that g_n will also be homotopic to a homeomorphism.

So we assume that γ_n is strictly increasing on $[0, a_n]$. Since $g_n \circ \varphi$ is constant on $[a_n, b_n]$, it follows that the same is true for γ_n. Next, we claim that γ_n must be strictly increasing on $[b_n, 1]$. Since $g_n \circ \varphi$ is 1–1 on this interval, the same must be true for γ_n, which means that the latter is either strictly increasing or decreasing on the interval. If it were decreasing, this would contradict the previously stated injectivity properties of g_n. Therefore γ_n is nondecreasing and nonconstant, so there is a positive integer d such that $\gamma_n(1) = t_n + d$. If $d = 1$ then it will follow that g_n is homotopic to the identity, so it is only necessary to show that d cannot be greater than 1. But if this were the case, then there would be some $s \in (0, 1)$ such that $\gamma_n(s) = 1$ and hence if $z_s = p(s)$, then $z_s \neq 1$ and $g_n(z_s) = g_n(1)$. However, by construction the function g_n is 1–1 off the image of the subinterval $[a_n, b_n]$ under φ, and this image does not contain $g_n(1)$. Hence we see that d must be equal to 1, and as noted before this proves the claim.

Returning to the proof of the theorem, let r_n denote the restriction of r_1 to $B_n \subset B_1$, so that the previous discussion implies that $r_n|C_n$ is a homotopy equivalence. It follows immediately that for each n the map r_n cannot be homotopic to a constant mapping; if this were so then $r|C_n$ would be homotopic to a constant and the same would be true of the associated homotopy equivalence g_n from S^1 to itself. Since no such map is homotopic to a constant, the assertion regarding r_n follows.

We shall now prove that $r|P$ also is not homotopic to a constant mapping. Assume the contrary, and let H be a homotopy from $r|P$ to a constant map. Extend H to a continuous map H' on $P \times B_1 \times [0, 1]$ by letting H' be given by r on $B_1 \times \{0\}$ and by the appropriate constant map on $B_1 \times \{1\}$. Now apply the Tietze Extension Theorem to construct a continuous extension of H' to a continuous map K_0 from $\mathbb{R}^2 \times [0, 1]$ to $\mathbb{R}^2$. Let W_0 denote the inverse image of $\mathbb{R}^2 - \{\mathbf{0}\}$ with respect to K_0. Then W_0 is an open neighborhood of $\{0\} \times [-1, 1] \times [0, 1]$, and by the Tube Lemma it contains a subset of the form

$$\left[0, \frac{1}{(2k+1)\pi}\right] \times [-1, 1] \times [0, 1]$$

for some positive integer k. If $\mathbf{U}$ is the usual retraction from $\mathbb{R}^2 - \{\mathbf{0}\}$ to S^1 which sends $\mathbf{v}$ to $|\mathbf{v}|^{-1}\,\mathbf{v}$, then on the set $B_k \times [0, 1]$ the map $K(x, t) = \mathbf{U}\big(K_0(x, t)\big)$ defines a homotopy from r_k to a constant map from B_k to S^1.

The preceding sentence contradicts our earlier conclusion that r_k is not homotopic to a constant; the source of the contradiction is our assumption that $r|P$ is homotopic to a constant, and therefore this must be false and the assertion in the theorem must be true.■

Covering spaces of the Polish Circle

If a topological space X is Hausdorff, simply connected, and locally arcwise connected, then the results of this course imply that every connected covering space over X is 1-sheeted (and hence a homeomorphism). In contrast, we have the following result:

PROPOSITION 6. *If P is the Polish Circle and n is either a positive integer or ∞, then there is an n-sheeted covering space projection $p_n : X_n \to P$ where X is connected.*

Sketch of proof. We shall give an explicit construction; in some cases, the verification of a particular property of this construction will be left to the reader as an exercise.

Assume first that $n = \infty$. The first step is to decompose P as a union of two closed subsets $K \cup L$, where L is the intersection of the line $\{ \ y \ = \ -2 \ \}$ and K is the closure of the complement of L (in terms of the description of P in `polishcircleA.pdf`, L is the bottom segment and K is the union of the vertical segments and the portion of the graph of $y = \sin(1/x)$ which lies between $x = 0$ and $x = 1$). Both K and L are connected, and their intersection consists of the two points $u = (0, -2)$ and $v = (1, -2)$. Then Y_∞ can be constructed by taking the disjoint union of $K \times \mathbb{Z}$ and $L \times \mathbb{Z}$ modulo identifying (u, k) in $K \times \mathbb{Z}$ with its counterpart (u, k) in $L \times \mathbb{Z}$ for all $k \in \mathbb{Z}$, and identifying (v, k) in $L \times \mathbb{Z}$ with $(v, k+1)$ in $K \times \mathbb{Z}$; the candidate for a covering space projection $p_\infty : Y_\infty \to P$ is then obtained by projecting onto the first coordinates of $K \times \mathbb{Z}$ and $L \times \mathbb{Z}$, and the inverse image of a point w in P is given by the (equivalence classes of the) points (w, k) where k runs through all the integers, and there are infinitely many such points.

To prove that p_∞ is a covering space projection, we need to show that every point has an evenly covered neighborhood. For all points except u and v, this follows immediately from the construction, in which both $P - K$ and $P - L$ are evenly covered. To dispose of the remaining two points, let M and N denote the vertical line segments in P such that u and v are endpoints of M and N respectively, and let M_0 and N_0 be the half open intervals $\{0\} \times [-2, -1]$ and $\{1\} \times [-2, -1]$. Then the construction of Y_∞ implies that $M_0 \cup L \cup N_0$ is an evenly covered open neighborhood of both u and v. image of

If n is finite, then a similar construction yields an n-sheeted covering if we replace $K \times \mathbb{Z}$ and $L \times \mathbb{Z}$ with $K \times \mathbb{Z}_n$ and $L \times \mathbb{Z}_n$ respectively.∎

Note that the covering space constructed in Proposition 6 are not arcwise connected; in fact, there is a 1–1 correspondence between the arc components of the covering space and the number of sheets in the covering.

Addendum: The Bruschlinsky group

The proof that P is not contractible involved the construction of a continuous map $P \to S^1$ that is not homotopic to a constant map. In this course sequence one basic topic is the fundamental group, which is given by basepoint preserving homotopy classes of continuous maps from S^1 to a space X; dually, one can develop an introductory topology course using a group of homotopy classes of continuous maps from a space X to S^1. We shall explain briefly how this can be done; a more complete development of this theme appears in the book, *A Geometric Introduction to Topology*, by C. T. C. Wall (Dover Books, NYC, 2011; ISBN 0486678504).

The dual group described in the preceding paragraph is called the *Bruschlinsky group*. Formally, if we are given a nonempty topological space X, define $\pi^1(X)$ to be the set of all homotopy classes $[X, S^1]$. Previous results and exercises show that the canonical map from $\pi_1(S^1, 1)$ to $\pi^1(S^1)$ is an isomorphism. By one of the exercises, the group structure on $\pi_1(S^1, 1)$ can be defined by taking the pointwise product of two closed curves in S^1, and more generally pointwise multiplication defines an abelian group structure on $\pi^1(X)$; specifically, if u and v are represented by $f : X \to S^1$ and $g : X \to S^1$, then $u \cdot v$ is represented by their pointwise product $f \cdot g$ (a little work is needed to show that the product is continuous and its homotopy class depends only on the homotopy classes of f and g). This construction has several properties that are analogous to those of the fundamental group.

(1) *If* $h : Y \to X$ *is a continuous map, then there is a homomorphism* $h^* : \pi^1(X) \to \pi^1(Y)$ *such that* h^* *takes the homotopy class of* $f : X \to S^1$ *to the homotopy class of* $f \circ h$*.*

(2) *In the preceding construction, if* h_0 *and* h_1 *are homotopic maps, then* $h_0^* = h_1^*$*.*

(3) *If* h *is the identity map on* X*, then* h^* *is the identity homomorphism. If* $k : W \to Y$ *is another continuous map, then* $(h \circ k)^* = k^* \circ h^*$*.*

(4) *If* h *is a constant map, then* h^* *is the trivial homomorphism.*

(5) *If "*$\amalg$*" denotes the disjoint union, then there is a canonical isomorphism from the group* $\pi^1\big(X \amalg Y\big)$ *to* $\pi^1(X) \times \pi_1(Y)$ *such that the algebraic coordinate projections correspond to the homomorphisms induced by the standard inclusions of* X *and* Y *in* $X \amalg Y$*.*

A more detailed discussion of this group appears in Section II.7 (= pp. 47–52) of Hu, *Homotopy Theory* (Academic Press, New York, 1959).