Pushout diagrams

Given a diagram of group homomorphisms

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] & G_1 \\ G_2 & \end{tikzcd}$$

a *pushout* of $\begin{tikzcd} H \arrow[r] \arrow[d] & G_1 \\ G_2 & \end{tikzcd}$ is a diagram

$$\begin{tikzcd} & G_1 \arrow[d, "h_1"] \\ G_2 \arrow[r, "h_2"] & P \end{tikzcd}$$

such that

① $h_1 f_1 = h_2 f_2$

② (Universal mapping property) Given any commutative diagram

$$\begin{tikzcd} H \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, phantom, "="] & G_1 \arrow[d, "\theta_1"] \\ G_2 \arrow[r, "\theta_2"] & Q \end{tikzcd}$$

with $\theta_1 f_1 = \theta_2 f_2$

there is a unique hom. $k : P \to Q$ such that $k h_1 = \theta_1$, $k h_2 = \theta_2$

$$\begin{tikzcd}
H \arrow[rr, "f_1"] \arrow[dd, "f_2"'] & & G_1 \arrow[dl, "h_1"'] \arrow[dd, "\theta_1"] \\
& P \arrow[dr, "k"] & \\
G_2 \arrow[ur, "h_2"] \arrow[rr, "\theta_2"'] & & Q
\end{tikzcd}$$

$G_1 * G_2$

<u>Examples.</u> ① $H = \{1\}$, get <u>free product</u>

② f_1, f_2 both 1-1, get <u>amalgamated free product</u>.

$G_1 *_H G_2$.

③ f_1 bijective $\Rightarrow P \cong G_2$

<u>Main Results</u> ① Pushouts always exist.

② If
$\begin{tikzcd} & G_1 \arrow[d, "h_1"] \\ G_2 \arrow[r, "h_2"] & P \end{tikzcd}$
and
$\begin{tikzcd} & G_1 \arrow[d, "h_1'"] \\ G_2 \arrow[r, "h_2'"] & P' \end{tikzcd}$
are pushouts,

then there is a unique isomorphism $\varphi : P \to P'$
such that $\varphi h_1 = h_1'$, $\varphi h_2 = h_2'$.

<u>Proofs</u> ① Start with $G_1 * G_2$ and the inclusions $j_i : G_i \subseteq G_1 * G_2$. Let N be the normal subgroup normally generated by all elements of the form $j_1 f_1(h) \cdot j_2 f_2(h)^{-1}$ where $h \in H$ is arbitrary, and take $P = G_1 * G_2 / N$ with $h_i = \overset{q}{(\text{quotient projection})} \circ j_i$ for $i = 1, 2$. It is elementary to check that $h_1 f_1 = h_2 f_2$.

To prove the universal mapping property, suppose $\theta_i : G_i \to Q$ satisfy $\theta_1 f_1 = \theta_2 f_2$. Then there is a unique homomorphism $\bar{k} : G_1 * G_2 \to Q$ s.t. $\bar{k} \circ j_i = \theta_i$. <u>CLAIM</u>: $\bar{k}|N$ is trivial. If so, <u>passage</u> to quotients yields a homomorphism $k : P \to Q$ s.t. $\bar{k} = k \circ q$, and $k \circ h_i = k \circ q \circ j_i = \bar{k} \circ j_i = \theta_i$, as desired. To prove $\bar{k}|N$ is trivial, it suffices to prove this for <u>the</u> normal generators $j_1 f_1(h) j_2 f_2(h)^{-1}$. Apply $\bar{k}$ to this element; then we get $\bar{k} j_1 f_1(h) \bar{k} j_2 f_2(h)^{-1} =$

$\theta_1 \cancel{f_1}\, f_1(h) \cdot \theta_2 f_2(h)^{-1}$ and this is 1 because $\theta_1 f_1 = \theta_2 f_2$. This proves the existence half of the universal mapping property.

Finally, to prove uniqueness, suppose that $k, k' : P \to Q$ satisfy $k h_i = \theta_i = k' h_i$. This means $k = k'$ on the images of G_1 and G_2 in P. Since these images generate P, it follows that $k = k'$ on all of P. ∎

(2) We have homomorphisms $\begin{Bmatrix} \alpha : P \to P' \\ \beta : P' \to P \end{Bmatrix}$ such that $\begin{Bmatrix} \alpha h_i = h_i' \\ \beta h_i' = h_i \end{Bmatrix}$. These imply

$h_i = \beta\alpha h_i$, $h_i' = \alpha\beta h_i'$. But we also have $h_i = 1_P h_i$, $h_i' = 1_{P'} h_i'$, so by uniqueness $\beta\alpha = 1_P$ and $\alpha\beta = 1_{P'}$; hence α and β are isomorphisms. Take $\varphi = \alpha$. ∎