

Pushout diagrams

Given a diagram of group homomorphisms

$$\begin{array}{ccc} H & \xrightarrow{f_1} & G_1 \\ f_2 \downarrow & & \\ G_2 & & \end{array}$$

a pushout of $\begin{array}{ccc} H & \rightarrow & G_1 \\ & \downarrow & \\ & G_2 & \end{array}$ is a diagram

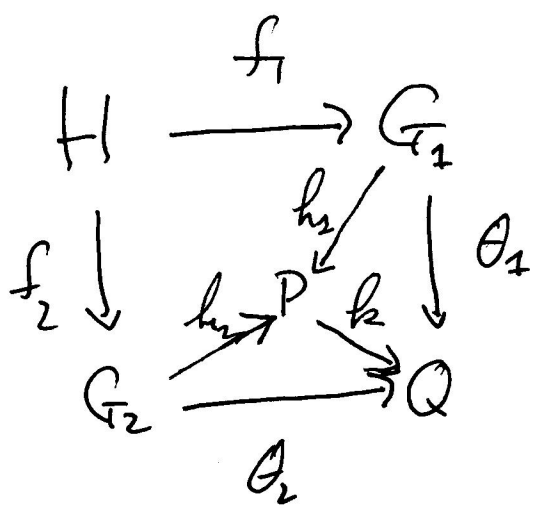
$$\begin{array}{ccc} & G_1 & \\ & \downarrow h_1 & \\ G_2 & \xrightarrow{h_2} & P \end{array} \quad \text{such that} \quad \textcircled{1} \quad h_1 f_1 = h_2 f_2$$

② (Universal mapping property) Given any commutative diagram

$$\begin{array}{ccc} H & \xrightarrow{f_1} & G_1 \\ f_2 \downarrow & \cong & \downarrow \theta_1 \\ G_2 & \xrightarrow{f_2} & Q \end{array} \quad \text{with} \quad \theta_1 f_1 = \theta_2 f_2$$

there is a unique hom. $k: P \rightarrow Q$ such that $kh_1 = \theta_1$, $kh_2 = \theta_2$

(2)



Examples. ① $H = \{1\}$, get free product $G_1 * G_2$

② f_1, f_2 both 1-1, get amalgamated free product.

$$G_1 *_H G_2.$$

③ f_1 bijective $\Rightarrow P \cong G_2$

Main Results ① Pushouts always exist.

② If $\begin{array}{ccc} & G_2 & \\ & \downarrow h_1 & \\ G_2 & \xrightarrow{h_2} & P \end{array}$ and $\begin{array}{ccc} & G_1 & \\ & \downarrow h_1' & \\ G_2 & \xrightarrow{h_2'} & P' \end{array}$ are pushouts,

then there is a unique isomorphism $\varphi: P \rightarrow P'$ such that $\varphi h_1 = h_1'$, $\varphi h_2 = h_2'$.

(3)

Proofs ① Start with $G_1 * G_2$ and inclusions $j_i: G_i \subseteq G_1 * G_2$. Let N be the normal subgroup normally generated by all elements of the form $j_1 f_1(h) \cdot j_2 f_2(h)^{-1}$ where $h \in H$ is arbitrary, and take $P = G_1 * G_2 / N$ with $h_i = (\text{quotient projection}) \circ j_i$ for $i=1, 2$. It is elementary to check that $h_1 f_1 = h_2 f_2$.

To prove the universal mapping property, suppose $\theta_i: G_i \rightarrow Q$ satisfy $\theta_1 f_1 = \theta_2 f_2$. Then there is a unique homomorphism $\bar{k}: G_1 * G_2 \rightarrow Q$ s.t. $\bar{k} \circ j_i = \theta_i$. CLAIM: $\bar{k}|_N$ is trivial. If so, passage to quotient yields a homomorphism $k: P \rightarrow Q$ s.t. $k = \bar{k} \circ q$, and ~~$\bar{k} \circ h_i = k \circ \theta_i$~~ $k \circ h_i = k \circ q \circ j_i = \bar{k} \circ j_i = \theta_i$, as desired. ~~To prove~~ To prove $\bar{k}|_N$ is trivial, it suffices to prove this for the normal generators $j_1 f_1(h) j_2 f_2(h)^{-1}$. Apply \bar{k} to this element; then we get $\bar{k} j_1 f_1(h) \bar{k} j_2 f_2(h)^{-1} =$

$\theta_1 f_1(h) \cdot \theta_2 f_2(h)^{-1}$ and this is 1

because $\theta_1 f_1 = \theta_2 f_2$. This proves the existence half of the universal mapping property.

Finally, to prove uniqueness, suppose that $k, k' : P \rightarrow Q$ satisfy $k h_i = \theta_i = k' h_i$. This means $k = k'$ on the images of G_1 and G_2 in P . Since these images generate P , it follows that $k = k'$ on all of P . \square

(2) We have homomorphisms $\left\{ \begin{matrix} \alpha : P \rightarrow P' \\ \beta : P' \rightarrow P \end{matrix} \right\}$

such that $\left\{ \begin{matrix} \alpha h_i = h'_i \\ \beta h'_i = h_i \end{matrix} \right\}$. These imply

$h_i = \beta \alpha h_i, h'_i = \alpha \beta h'_i$. But we also have $h_i = 1_P h_i, h'_i = 1_{P'} h'_i$, so by uniqueness

$\beta \alpha = 1_P$ and $\alpha \beta = 1_{P'}$; hence α and β are isomorphisms. Take $\varphi = \alpha$. \square