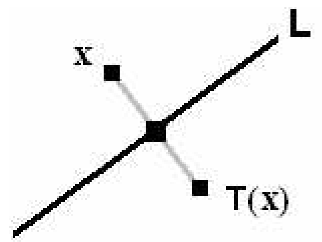
NOTE ON REFLECTIONS

One particularly important class of isometries for $\mathbb{R}^2$ is given by **plane reflections**. Geometrically, such maps may be described as follows: There is a line L (the **axis of symmetry**) such that every point in L is sent to itself, but if $\mathbf{x} \notin L$ then $\mathbf{x}$ and its image $\mathbf{x}'$ are distinct, and the line L is the perpendicular bisector of the segment $[\mathbf{x}\mathbf{x}']$.



The purpose of this note is to give a formula for such a mapping in terms of linear algebra.

Write $L = \mathbf{q} + V$, where $\mathbf{q} \in L$ and V is a 1-dimensional vector subspace of $\mathbb{R}^2$, and let $\{\mathbf{v}\}$ be an orthonormal basis for V.

LEMMA. *Let A be the linear mapping of $\mathbb{R}^2$ given by $A\mathbf{x} = 2 \cdot \langle \mathbf{x}, \mathbf{v} \rangle\, \mathbf{v} - \mathbf{x}$. Then $A\mathbf{x} = \mathbf{x}$ if $\mathbf{x} \in V$ and $A\mathbf{x} = -\mathbf{x}$ if $\mathbf{x}$ is perpendicular to $\mathbf{v}$.*

Proof. Before starting, we note that linearity is an immediate consequence of the definition and properties of the inner product $\langle\ ,\ \rangle$.

If $\mathbf{x} \in V$, then $\mathbf{x} = c\mathbf{v}$ for some scalar c, and

$$A\mathbf{x} \;=\; A(c\mathbf{v}) \;=\; c(A\mathbf{v}) \;=$$

$$c\big(\, 2\langle \mathbf{v}, \mathbf{v} \rangle \;-\; \mathbf{v}\big) \;=\; c(2 \cdot \mathbf{v} \;-\; \mathbf{v}) \;=\; c\mathbf{v} \;=\; \mathbf{x}\ .$$

On the other hand, if $\mathbf{x}$ is perpendicular to V, then $\langle \mathbf{x}, \mathbf{v} \rangle = 0$ and therefore we have $A\mathbf{x} = -\mathbf{x}$.■

The preceding lemma leads directly to an purely algebraic formula for the reflection map that we have described in geometric terms.

PROPOSITION. *Let $L = \mathbf{q} + V$ and A be as above, and let $T(\mathbf{x}) = A\mathbf{x} + (\mathbf{q} - A\mathbf{q})$. Then the following hold:*

(i) $T(\mathbf{x}) = \mathbf{x}$ if $\mathbf{x} \in L$.

(ii) For all $\mathbf{x} \notin L$ we have $T(\mathbf{x}) \neq \mathbf{x}$.

(iii) If $\mathbf{x} \notin L$ then the line joining $T(\mathbf{x})$ and $\mathbf{x}$ meets L at the midpoint of the segment $[\mathbf{x}\,T(\mathbf{x})\,]$, and the lines L and $\mathbf{x}\,T(\mathbf{x})$ are perpendicular.

Since T has the geometrical propeties of the reflection map in the previous discussion, it follows that T must be the reflection map we described above.

Proof. (i) If $\mathbf{x} \in L$ then $\mathbf{x} = \mathbf{q} + b\mathbf{v}$ for some scalar b. Then we have

$$T(\mathbf{x}) \;=\; T(\mathbf{q} + b\mathbf{v}) \;=\; A(\mathbf{q} + b\mathbf{v}) + \mathbf{q} - A\mathbf{q} \;=$$

$$A\mathbf{q} + b\, A\mathbf{v} + \mathbf{q} - A\mathbf{q} \;=\; b(A\mathbf{v}) + q$$

and since $A\mathbf{v} = \mathbf{v}$ this implies that $T(\mathbf{x}) = \mathbf{q} + b\mathbf{v} = \mathbf{x}$.

(ii) If $\mathbf{x} \notin L$ then $\mathbf{x} - \mathbf{q} = \mathbf{y} + \mathbf{z}$ where $\mathbf{y} = b\mathbf{v}$ for some scalar b and $\mathbf{z}$ is a nonzero vector which is perpendicular to V. Therefore we have

$$T(\mathbf{x}) \;=\; A(\mathbf{q} + b\mathbf{v} + \mathbf{z}) + \mathbf{q} - A\mathbf{q} \;=$$

$$A\mathbf{q} + A(b\mathbf{v}) + A\mathbf{z} + \mathbf{q} - A\mathbf{q} \;=\; A\mathbf{q} + b\mathbf{v} - \mathbf{z} + \mathbf{q} - A\mathbf{q} \;=\; \mathbf{q} + b\mathbf{v} - \mathbf{z}\ .$$

It follows that $x - T(\mathbf{x}) = 2 \cdot \mathbf{z}$ and is nonzero, so that $\mathbf{x} \neq T(\mathbf{x})$.

(iii) Continuing in the setting of the previous part, we see that the line $\mathbf{x}\,T(\mathbf{x})$ is equal to

$$\mathbf{x} \;+\; \mathbb{R} \cdot \big(\mathbf{x} - T(\mathbf{x})\big) \;=\; \mathbf{x} \;+\; \mathbb{R} \cdot (2\mathbf{z}) \;=\; x \;+\; \mathbb{R} \cdot \mathbf{z}$$

(since $\mathbb{R} \cdot (2\mathbf{z}) = \mathbb{R} \cdot \mathbf{z}$), and this line is perpendicular to L because $\mathbf{z}$ is perpendicular to $\mathbf{v}$. Finally, we have $\frac{1}{2}\big(\mathbf{x} + T(\mathbf{x})\big) = \mathbf{q} + b\mathbf{v}$ which is also a point of L, and hence L meets $\mathbf{x}\,T(\mathbf{x})$ at the midpoint of $[\mathbf{x}\,T(\mathbf{x})\,]$.■