

## Addendum to Section VIII.2

The fundamental group is very useful for showing that certain pairs of spaces are not homeomorphic.

**THEOREM 11.** *Let  $X$  and  $Y$  be arcwise connected topological spaces with  $p \in X$  and  $q \in Y$ . If  $X$  and  $Y$  are homeomorphic, then  $\pi_1(X, p)$  is isomorphic to  $\pi_1(Y, q)$ .*

In particular, if the fundamental groups of  $X$  and  $Y$  are not isomorphic, then  $X$  and  $Y$  cannot be homeomorphic.

**Proof.** Suppose that  $f : X \rightarrow Y$  is a homeomorphism, and let  $g = f^{-1}$ . Then by the functoriality of induced homomorphisms for fundamental groups we know that  $f_* : \pi_1(X, p) \rightarrow \pi_1(Y, f(p))$  is an isomorphism whose inverse is  $g_* : \pi_1(Y, f(p)) \rightarrow \pi_1(X, p)$ . By Theorem 6 we know that  $\pi_1(Y, f(p))$  is isomorphic to  $\pi_1(Y, q)$ , and therefore we also have that  $\pi_1(X, p)$  is isomorphic to  $\pi_1(Y, q)$ . ■