$E \to X$ covering
$z \in X \Rightarrow$ Fiber of z
$=$ inverse image of $\{z\}$

Exercises 02 #5

~~All~~ Two parts of this exercise depend upon the following:

<u>Theorem</u> The set of arc components of $E|A$ is in 1-1 correspondence with the cosets $\pi_1(X) / f_*[\pi_1(A)]$.

<u>Proof.</u> Choose a base point $z \in A \subseteq X$.

(1) Suppose $u, v \in E|A$ map to z, and write them as $w \cdot a$, $w \cdot b$ (*) for a suitable base point in $E|A$ which maps to z.

(*) $a, b \in \pi_1(X)$

If $w \cdot a$ and $w \cdot b$ ~~map to~~ are in the same component of $E|A$, then there is a curve joining them, and it projects to some $c \in \pi_1(A)$. By the construction of the $\pi_1(X)$ action on the fiber of z <u>in X</u>, it follows that $w \cdot b = w \cdot a \cdot f_*(c)$, so a and b go to the same class in $\pi_1(X) / f_*[\pi_1(A)]$:

(2) Conversely, suppose that $a = b f_*(c)$ for some $c \in \pi_1(A)$. Then by path lifting we know that $w \cdot a$ and $w \cdot b$ lie in the same component of $E|A$. ∎

Application to (i)–(iii)

(i) The ~~[illegible]~~ components of $E|A$ are in 1-1 correspondence with $\pi_1(X)$, and each component contains a unique point in the fiber of z. Hence the restriction of $E|A$ to each component is 1-sheeted.

(ii) The set of components is $\pi_1(X)/f_*[\pi_1(A)]$, which is one point if f_* is onto. Hence $E|A$ is connected.

(iii) Pick a component C of $E|A$ and a base point $w \in C$ which maps to z. Let $\gamma: I \longrightarrow C$ be a basepoint preserving closed curve. Then $\gamma \simeq$ constant in E, and we have

$$\begin{tikzcd}
\pi_1(C) \arrow[r] \arrow[d, "p'_*"'] \arrow[dr, phantom, "="] & \pi_1(E) = \{1\} \arrow[d, "p_*"] \\
\pi_1(A) \arrow[r, "f_*", "1-1"'] & \pi_1(X)
\end{tikzcd} \qquad p' = p|C$$

From the diagram it follows that $f_* p'_* [\gamma]$ is trivial. Since f_* and p'_* are 1-1, it follows that $[\gamma]$ is trivial. ∎

Exercises 0 2 #8

This uses the formula for finding the number of generators of $\pi_1(X)$ if $X \to Y$ is a finite covering of graph complexes.

$F_2 = \pi_1\Big($  $\Big)$ free gens x, y.

graph

If $H_3 = \text{Kernel } F_2 \longrightarrow \mathbb{Z}_2$ the formula

$x \longrightarrow 0$

$y \longrightarrow 1$

implies H_3 is free on 3 generators.

Suppose we have constructed

$H_k \subseteq \ldots \subseteq H_3$; we need to

construct H_{k+1}. Take $K \subseteq H_{k+1}$ to be a free group on two free generators (just pick 2 from some set of k free gens. for H_k), and let $K_0 \subseteq K$ be the index 2 subgp with 3 generators. Finally, take H_{k+1} to be the group generated by K_0 and the extra generators of H_k. If K_1 = free subgp gen by the generators, then we have

$$H_{k+1} \cong K_0 * K_1 \text{ (free product)}$$

$$\subseteq K * K_1 = H_k,\ \text{so}$$

H_{k+1} is free on $k+1$ generators.