More Solutions for Exercises 03

12. (a) Let $i: A \to B$ and $j: B \to C$. We need to show

① $0 \to \mathrm{Hom}(G,A) \xrightarrow{i_*} \mathrm{Hom}(G,B)$ is exact (so i_* is 1-1)

② $\mathrm{Hom}(G,A) \xrightarrow{i_*} \mathrm{Hom}(G,B) \xrightarrow{j_*} \mathrm{Hom}(G,C)$ is exact.

① Let $f: G \to A$; we need to show that $i_* f = i \circ f = 0 \Rightarrow f = 0$. This follows because if $g \in G$, then $i_* f(g) = 0$ all $G \Rightarrow 0 = i \circ f(g)$ all g, and since i is 1-1 this means $f(g) = 0$ all g, so $f = 0$. ■

② $j_* \circ i_* = (j \circ i)_* = 0_* = 0 \Rightarrow$ Image $i_* \subseteq$ Kernel j_*. Conversely, if $h: G \to B$ and

$j_* h = j \circ h = 0$, then $g \in G \Rightarrow j \circ h(g) = 0$, so $h(g) \in$ Kernel j all g, so that $h(g) \in A$ all g. Hence $h = i \circ h_0$ for some $h_0: G \to A$. ■

$\boxed{(b)}$ We use the same notation and a similar approach

1. Show ~~Hom(G, B) →~~
$$0 \to Hom(C, G) \xrightarrow{j^*} Hom(B, G) \text{ is 1-1}$$

2. $$Hom(C, G) \xrightarrow{j^*} Hom(B, G) \xrightarrow{i^*} Hom(A, G)$$

is exact.

1. Suppose $f: C \to G$ and $f \circ j = 0$. If $c \in C$ then $c = j(b)$ for some b and hence $f(c) = f \circ j(b) = 0$, so that $f = 0$. ∎

2. $i^* \circ j^* = (j \circ i)^* = 0^* = 0$, so Image $j^* \subseteq$ Kernel i^*. Conversely,

$i^* f = 0 \Rightarrow f|A = 0 \Rightarrow$ we can write

$f = \bar{f} \circ j$ for a unique $\bar{f}: C \cong B/A \to G$. ∎

$= j^* \bar{f}$.

By (a) and (b)

$\boxed{(c)}$ We need only show that
$$Hom(G, A \oplus C) \longrightarrow Hom(G, C)$$
$$Hom(A \oplus C, G) \longrightarrow Hom(A, G)$$
are <u>onto</u>.

The quickest way to dispose of these is as follows. Given $f: G \to C$, let $F: G \to A \oplus C$ be $F(g) = (0, f(g))$ and given $f: A \to G$, let $F: A \oplus C \to G$ be $F(a, c) = f(a)$. In the first case, $j_* F = f$, and in the second $i^* F = f$. ∎

13. Follow the hint. By construction the equivalence relation is generated by the binary relation $A \sim B \Leftrightarrow A \cap B \neq \emptyset$. Let $\mathcal{F}$ be an equivalence class of simplices, and note that $\mathcal{F}$ is closed under taking faces. Then $K(\mathcal{F})$ is a subcomplex. If $\mathcal{F} \neq \mathcal{F}'$ then $K(\mathcal{F}) \cap K(\mathcal{F}') = \emptyset$, so K is the union of the pairwise disjoint subcomplexes $K(\mathcal{F})$.

The underlying subspaces $P(\mathcal{F})$ decompose P = space of K into finitely many nonempty pairwise disjoint closed connected subspaces. This number of subspaces is $1 \Leftrightarrow P$ is connected. ∎

14. Let J = all pairs (E, T) where T is a 2-simplex and E is an edge of T. Let e = # edges, t = # 2-simplices.

Since each T has 3 edges, we have $|J| = 3t$, and the assumption implies that $|J| = 2e$. Hence $2e = 3t$, which means that e must be divisible by 3. ∎