Second Supplement to Chapter 5 of Sutherland,

Introduction to Metric and Topological Spaces (Second Edition)

Numerous examples of metric spaces are described on pp. 40 – 47 of Sutherland. Some other important examples arise naturally in synthetic geometry as presented in courses such as Mathematics 133 (for example, see **http://math.ucr.edu/~res/math133**).

Roughly speaking, a ***neutral plane*** is geometrical system in which all but the last of the classical postulates for Euclidean geometry (as in, say, Euclid's ***Elements***) are valid. More precisely, the data for a neutral plane are given by an ordered data $(\mathbf{P}, \mathcal{L}, d, \mu)$ where $\mathbf{P}$ is a set (the ***plane***) whose elements are called ***points***, $\mathcal{L}$ is a family of nonempty proper subsets of $\mathbf{P}$ called ***lines***, the ***distance function*** $d(\mathbf{X}, \mathbf{Y})$ associates a nonnegative real number to each pair of points $\mathbf{X}, \mathbf{Y}$ in the plane, and the ***angle measurement function*** μ associates an angle measure to each angle in the plane. An index to the assumed properties of these data appears on page 3 of **http://math.ucr.edu/~res/math133/foundations02.pdf** (specifically, a neutral plane satisfies the axioms in groups 1 through 6), and more specific descriptions of the primitive concepts and their assumed properties appear in the following documents from the previously cited directory:

http://math.ucr.edu/~res/math133/geometrynotes02a.f13.pdf

http://math.ucr.edu/~res/math133/geometrynotes02b.f13.pdf

As noted on the first two pages of **http://math.ucr.edu/~res/math133/neutral-proofs3.pdf**, the distance function on $\mathbf{P}$ satisfies the defining conditions for a metric space; in fact, the name of one defining conditions for a metric space (the Triangle Inequality) comes directly from a basic result in elementary geometry. However, the ties between classical geometry and metric spaces are much deeper than the formal definitions, for many types of geometrical figures and regions are either closed or open subsets of $\mathbf{P}$ with respect to the given metric. In particular, half – planes and interiors of angles are open subsets, and lines, closed rays and closed segments are closed subsets. Analogous statements hold for a neutral 3 – space $(\mathbf{S}, \mathcal{P}, \mathcal{L}, d, \mu)$ which have a family of subsets called ***planes*** as additional data and satisfy a slightly longer list of properties.

The preceding discussion suggests that concepts from the theory of metric spaces and continuous functions may be useful in studying certain issues arising from classical geometry, and the results in the online files **http://math.ucr.edu/~res/math133/convex-functions.pdf** and **http://math.ucr.edu/~res/math133/neutral-proofs3.pdf** are examples of this sort. Of course, one goal of this course is to study some other notions from elementary geometry like the concept of a region in the plane or in space. Calculus (in one or several variables) provides still further examples at a somewhat higher level, and the applications of calculus lead naturally to ***differential geometry***, which begins with the systematic application of calculus to study curves and surfaces. There are numerous references for the latter; in particular, the following takes a fairly classical approach, only assuming very basic material from multivariable calculus:

M. Lipschutz, *Schaum's Outlines — Differential Geometry* (Second Edition). Schaum's/McGraw – Hill, New York, NY, 1969.

The online directory **http://math.ucr.edu/~res/math138A-2012/** is based upon this reference, and its files contain further information.

Topological approaches to the foundations of geometry

In the preceding discussion, standard axioms for classical geometry produce examples of topological spaces. Conversely, it is also possible to develop foundations for geometry by starting with topological spaces and adding assumptions that they have additional data (for example, a family of nonempty closed subsets corresponding to lines, with other families corresponding to planes, k – planes and so on) and which have appropriate properties (for example, two distinct points lie on a unique line). There have been several studies in this direction, but we shall only mention one pair of papers in which the necessary mathematical background does not go beyond topics covered in standard undergraduate courses for mathematics majors.

M. C. Gemignani, *Topological geometries and a new characterization of* $\mathbb{R}^n$. Notre Dame Journal of Formal Logic **7** (1966), 57 – 100.

M. C. Gemignani, *On removing an unwanted axiom in the characterization of* $\mathbb{R}^m$ *using topological geometries*. Notre Dame Journal of Formal Logic **7** (1966), 365 – 366.

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