Mathematics 205B, Winter 2019, Assignment 1

Solutions for Assignment 1, Winter 2019

1. Let X be a space which is Hausdorff, connected and locally arcwise connected, let $f : (X, x) \to (S^1, 1)$ be a continuous mapping, and let $p : \mathbb{R} \to S^1$ be the map $p(t) = \exp(2\pi i t)$. Prove that f lifts to a map $F : X \to \mathbb{R}$ if and only if f is basepoint preservingly homotopic to a constant mapping.

SOLUTION

If f lifts to F, then F is homotopic to a constant map because $\mathbb{R}$ is contractible, and therefore $p \circ F = f$ is also homotopic to a constant map. Conversely, if f is homotopic to a constant map then the mapping of fundamental groups $f_* : \pi_1(X) \to \pi_1(S^1)$ is the trivial homomorphism, and hence its iimage is the trivial subgroup. We can now apply the Lifting Criterion to conclude that f lifts to a map $F : X \to \mathbb{R}$.

2. Let X be a space which is Hausdorff, connected and locally arcwise connected, let $x \in X$, and assume further that $\pi_1(X, x)$ is finite of order n. Prove that, up to equivalence, there are only finitely many connected covering spaces of X, and give an upper bound $B(n)$ for the number of equivalence classes of coverings as an explicit, elementary function of n.

SOLUTION

We know that the equivalence classes of connected coverings are in 1–1 correspondence with subgroups of the fundamental group. Since the latter has only finitely many elements, there are only finitely many possible subgroups, and hence there are only finitely many connected covering spaces of X. We can refine this further by estimating the number of subgroups in a group with n elements. Since there are 2^n subsets of a set with n elements, there can be at most 2^n subgroups. In fact, one can sharpen this to 2^{n-1} because we know that a subgroup must contain the identity element, so the number of possible subgroups in a finite group G is at most the number of subsets of $G - \{1\}$. This turns out to be the best possible estimate of the form 2^k because there are two subgroups of a group with two elements. Of course, for more general values of n this estimate is far from optimal; for example, a subroup of prime order always has exactly two subgroups.