Mathematics 205B, Winter 2018, Assignment 1

This will be due on **Monday, March 19, 2018, at 9:00 A.M,** which is the beginning of the final examination. If you wish to use some version of TEX in writing up your answers, please feel free to do so. The plain TEX file for this document is in the course directory.

You must show the work behind or reasons for your answers.

1. (*a*) Consider a hollow octahedron in 3-space with vertices A, B, C, D, E, F and 2-simplex faces listed below (take the usual alphabetical ordering for the vertices):

$$ACD\ ,\ ADE\ ,\ AEF\ ,\ ACF\ ,\ BCD\ ,\ BDE\ ,\ BEF\ ,\ BCF$$

(It is highly recommended that you sketch this figure or construct a cardboard model!). It is known that this space is homeomorphic to S^2 and hence its 2-dimensional homology is isomorphic to $\mathbb{Z}$. An explicit generator for the simplicial homology group is given by a linear combination of the 2-simplices such that the coefficient of each generator is ± 1. Find the generator for which the coefficient of ACD is $+1$.

(*b*) The boundary of a triangular prism can be described as a polyhedron with vertices A, B, C, D, E, F and 2-simplex faces listed below (the conventions and suggestions of (*a*) also apply here):

$$ABC\ ,\ DEF\ ,\ ADE\ ,\ ABE\ ,\ BEF\ ,\ BCF\ ,\ CDF\ ,\ ACD$$

Once again the 2-dimensional homology is isomorphic to $\mathbb{Z}$. An explicit generator for the simplicial homology group is given by a linear combination of the 2-simplices such that the coefficient of each generator is ± 1. Find the generator for which the coefficient of DEF is $+1$.

2. (*a*) Let $\mathbb{R}^n_+ \subset \mathbb{R}^n$ be the set of points whose first coordinates are nonnegative, and let W be the open subset of points whose first coordinate is positive. By results in the lectures, if $p \in W$ then

$$H_q(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{p\}) \cong H_q(W, W - \{p\}) \cong H_q(\mathbb{R}^n, \mathbb{R}^n - \{p\})\ \cong\ \mathbb{Z}\ (q = n)\ ,\quad 0\ (\text{otherwise})\ .$$

Prove that if $p \in \{0\} \times \mathbb{R}^{n-1} \subset \mathbb{R}^n_+$, then $H_q(\mathbb{R}^n_+, \mathbb{R}^n_+ - \{p\}) \cong 0$ for all q. [*Hint:* Why is the pair $(\{1\} \times \mathbb{R}^{n-1}, \{1\} \times \mathbb{R}^{n-1})$ a strong deformation retract of the pair in the preceding sentence, and why is $H_q(X, X)$ always zero? How does the long exact homology sequence of a pair imply the latter?]

(*b*) Prove the following generalization of Proposition VII.1.4: *Let U and V be open subsets of a topological space X, and suppose that $y \in U$ and $z \in V$. Let $f : U \to V$ be a homeomorphism such that $f(y) = z$. Then the local homology groups $H_*(X, X - \{y\})$ and $H_*(X, X - \{z\})$ are isomorphic.*

(*c*) Consider the special case of (*b*) where $X = \mathbb{R}^n_+$, and let U, V, y, z, h be as given there. Prove that the first coordinate of y is zero if and only if the first coordinate of z is zero; *i.e.*, h takes boundary points of $\mathbb{R}^n_+$ to boundary points and nonboundary points to nonboundary points.